Coherence in quantum estimation

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The geometry of quantum states provides a unifying framework for estimation processes based on quantum probes, and it allows to derive the ultimate bounds of the achievable precision. We show a relation between the statistical distance between infinitesimally close quantum states and the second order variation of the coherence of the optimal measurement basis with respect to the state of the probe. In Quantum Phase Estimation protocols, this leads to identify coherence as the relevant resource that one has to engineer and control to optimize the estimation precision. Furthermore, the main object of the theory i.e., the Symmetric Logarithmic Derivative, in many cases allows to identify a proper factorization of the whole Hilbert space in two subsystems. The factorization allows: to discuss the role of coherence vs correlations in estimation protocols; to show how certain estimation processes can be completely or effectively described within a single-qubit subsystem; and to derive lower bounds for the scaling of the estimation precision with the number of probes used. We illustrate how the framework works for both noiseless and noisy estimation procedures, in particular those based on multi-qubit GHZ-states. Finally we succinctly analyze estimation protocols based on zero-temperature critical behavior. We identify the coherence that is at the heart of their efficiency, and we show how it exhibits the non-analyticities and scaling behavior proper of a large class of quantum phase transitions.

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I. INTRODUCTION

Precision in single parameter estimation processes can be strikingly enhanced with the use of quantum probes [1–3]. Therefore the search for new and increasingly efficient quantum estimation schemes is at the basis of the development of several technologies, and it is an arduous theoretical and experimental challenge [4, 5]. Two paradigmatic examples are Quantum Phase Estimation (QPE) and Criticality-Enhanced Quantum Estimation (CEQE). In the first case the goal is to determine the phase λ of a unitary evolution $e^{-i\lambda G}$ generated by a fixed operator G. QPE is essential for several applications such as interferometry [8–10], spectroscopy [11, 12], magnetic sensing [13–16] and atomic clocks [17, 18].CEQE instead exploits the critical behavior of systems undergoing a quantum phase transition (QPT) to drastically enhance the estimation precision of the parameter driving the transition; the latter is in general a dynamic parameter (such as a coupling constant) of a complex many-body quantum system [19–21]. In both cases the precision's ultimate bounds can be established by means of quantum estimation theory [4–7].

A fundamental open question is whether it is possible to identify a single relevant resource underlying the optimal efficiency of all these estimation tasks.

In answering this question one is led to consider different aspects. First of all the estimation processes are dynamical, and one may expect that rather than the static properties of the state probe, what matters is their dynamical change. Secondly, since the probes are quantum, one should focus on the prominent resources that distinguish quantum from classical systems: coherence and correlations. The choice of coherence is a natural and intuitive one: many estimation protocols are indeed interference experiments [8–10]. But which is the *relevant* coherence? And how to quantify the latter in a consistent and general way?

On the other hand, as for QPE, many authors have focused on quantum correlations [22–24]. In particular, entanglement has been often indicated as the key to achieve a better asymptotic scaling of the sensitivity with the number of probes used. However, the relevance of quantum correlations can be and has been questioned in different ways. For example, when the estimation process is not affected by any noise, protocols based on completely uncorrelated probes (e.g. multi-round single qubit protocols) are able to reach the same sensitivity achieved by protocols based on highly quantum correlated probes (such as GHZ multi-qubit states) [24, 25, 27–31]. This indicates that in the noiseless case quantum correlations are not an intrinsic prerequisite for efficient QPE. Furthermore, the

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presence of noise in the evolution of the system is in general highly detrimental for the estimation processes, and the sole presence of quantum correlations is not in general a sufficient condition to counteract its effects and achieve an enhancement of the estimation sensitivity [26, 30, 66]. Finally, there is a more conceptual difficulty in identifying quantum correlations as a key resource for estimation. Indeed, correlations are typically defined once a specific tensor product structure (TPS) i.e., a factorization of the Hilbert space $\mathcal{H}$ in subsystems, is chosen to describe the whole system. But in general there can be many inequivalent TPSs and correspondingly many different kinds of (multipartite) quantum correlations; unless the problem at hand allows to identify a specific TPS in a unique way, it is not clear which among the various possibilities should be the relevant one. A possible way of "ruling out" the relevance of certain (quantum) correlations could be the following. Suppose there is at least one partition of the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ such that the whole estimation procedure can be described within the subsystem $\mathcal{H}_A$ alone. Then one can trace out subsystem $\mathcal{H}_B$ and (quantum) correlations between $\mathcal{H}_A$ and $\mathcal{H}_B$ never show up in the description of the whole procedure. If the description of the process in the chosen TPS is equivalent to the original one, then one may argue that those bipartite correlations do not play any role for the estimation task, and one has therefore to look for other resources at the basis of the estimation sensitivity.

As for CEQE, on the other hand, the estimation protocols are based on the occurrence of zero temperature transitions in the underlying many-body system. While QPTs have been thoroughly characterized via (bipartite/multipartite) quantum correlations [32–36], when one focuses on the use of criticality for enhancing the estimation sensitivity the main concept of the theory is the information-geometric notion of statistical distinguishability between neighboring ground states, rather than correlations [19–21, 38].

The notion of statistical distinguishability is ubiquitous in all the above mentioned processes. Indeed, a unified description of both noiseless/noisy QPE tasks and QPTs is provided by the powerful mathematical language of information geometry [7, 38, 39] in terms of infinitesimal state discrimination. It is therefore natural to look for connections between the bounds on the sensitivity achievable in quantum estimation and some fundamental feature of the underlying quantum system within the information geometry framework. The main quantity that allows to connect geometry, estimation and QPTs is the Quantum Fisher Information (QFI). The latter on one hand is proportional to the statistical geometrical distance between neighboring quantum states, and on the other hand it provides, via the Quantum Cramer Rao theorem, the ultimate bounds on parameter estimation with quantum probes.

In this work, we explore the existing connection between QFI and a primary resource of quantum probes: *coherence* [41–51]. In particular, we find a relation between the QFI and the *curvature of the coherence of the measurement basis that gives the optimal discrimination.* Indeed, coherence is a basis dependent feature and we show that *the relevant basis is given by the eigenvectors of the main object of the Cramer-Rao approach to quantum estimation: the Symmetric Logarithmic Derivative (SLD) L.* The relation found allows in the first place to highlight a possibly new physical interpretation of the statistical-geometric distance between infinitesimally close pure and mixed quantum states. On the other hand, it allows to *identify and quantify the relevant resource that must be engineered, controlled, exploited and preserved in QPE and CEQE protocols in order to achieve the highest possible precision.* The main focus of our work will be QPE. In developing our theoretical framework, we will show how it can be applied to instances of both noiseless and noisy QPE protocols. In all treated cases the relation between coherence and QFI holds independently of any Hilbert space partition.However, we will argue that in many cases it is possible to select in a unique way a proper TPS tailored to the problem at hand. The relevant TPS (TPS^R) is again suggested by the Symmetric Logarithmic Derivative.The factorization of the Hilbert space induced by L allows to neatly examine different aspects of estimation protocols. In the first place it is possible to find a connection between the QFI and bipartite classical (rather than quantum) correlations between the set of observables that are relevant for the process [56]. Within our perspective the relation between the achievable estimation precision and coherence vs correlations can be easily discussed. While coherence is in general fundamental for the process, we will argue that in many cases quantum correlations such as entanglement and discord can be seen as irrelevant or even detrimental. Parallelly, we will show that upon adopting the TPS^R, in many relevant examples the whole estimation procedure can entirely or effectively be described only within a subsystem. In such cases, whatever the dimension of the original quantum system, the estimation process is seen to be equivalent to a single qubit (multi-round) one. Notable examples are procedures based on multi-qubit GHZ states or certain class of NOON states [57]. In particular we will discuss why highly entangled states, such as GHZ, can lead to a substantial enhancement of the estimation sensitivity even in presence of some kind of noisy processes. Finally, we will show that the description given by the TPS^R may allow to easily derive meaningful lower bounds on the scaling of the precision with the number of probes M used or with the dimension N of the Hilbert space.

In the last part of our work we will turn our attention to CEQE. Here we succinctly explore the consequences of the found link between information geometry and coherence for the fidelity approach to Quantum Phase Transitions, and for the estimation protocols based or zero-temperature criticality. We show that the non-analiticities that characterize the approach of a many-body system to a critical point, and that are at the basis of CEQE, can be interpreted as the divergent behavior of a specific coherence function.

The paper is organized as follows. In Section II we derive the relation between QFI and coherence. In Section III we apply the relation found to two- and N-dimensional systems and we introduce the TPS^R. With the aid of the latter we discuss the role of coherence vs (quantum) correlations, and we analyze specific relevant cases such as protocols based on GHZ and NOON states. In Section IV we discuss how our framework can be applied to noisy estimation processes. In particular we thoroughly discuss a relevant example based on multi-qubit GHZ states. In Section V we finally assess the role of coherence in CEQE. For the sake of clarity and conciseness, the details of the calculations leading to our main results can be found in the Appendices.

II. QFI AND COHERENCE.

The problem of identifying the ultimate precision in the estimation of a given parameter λ can be described within the Quantum Crámer-Rao formalism. In this context, an unknown $\lambda \in \mathbb{R}$ parametrizes a family of quantum states ρ_λ of an $N-$dimensional quantum system. Given any (unbiased) estimator $\hat{\lambda}$ of λ, the ultimate bound in terms of the variance of $\hat{\lambda}$ reads

$$\Delta^2\hat{\lambda} \geq (M\ QFI)^{-1} \tag{1}$$

where M is the number of independent copies of ρ_λ and

$$QFI\left(\rho_\lambda\right) = Tr\left[\rho_\lambda L_\lambda^2\right] \tag{2}$$

is the Quantum Fisher Information. The latter is expressed in terms of the Symmetric Logarithmic Derivative (SLD) L_λ, the Hermitian operator that satisfies the equation

$$\partial_\lambda\rho_\lambda = \left(\rho_\lambda L_\lambda + L_\lambda\rho_\lambda\right)/2 \tag{3}$$

The bound in general can be attained by implementing an experiment projecting the state onto the eigenbasis $\mathcal{B}_{\boldsymbol{\alpha}}^\lambda = \left\{|\alpha^\lambda\rangle\right\}_{\alpha=1}^N$ of L_λ. The QFI is the maximum of the Fisher Information $FI\left(\mathcal{B}_{\mathbf{x}};\rho_\lambda\right) = \sum_x\left(\partial_\lambda p_x^\lambda\right)^2/p_x^\lambda$ over all possible experiments (orthonormal bases) $\mathcal{B}_{\mathbf{x}} = \{|x\rangle\}_{x=1}^N$, where $p_x^\lambda = \langle x|\rho_\lambda|x\rangle$ is the probability of obtaining the outcome x:

$$QFI(\rho_\lambda) = \max_{\mathcal{B}_{\mathbf{x}}} FI\left(\mathcal{B}_{\mathbf{x}};\rho_\lambda\right) = FI\left(\mathcal{B}_{\boldsymbol{\alpha}}^\lambda;\rho_\lambda\right) \tag{4}$$

The above formalism is common to all single-parameter quantum estimation processes, and the QFI was shown to bear a fundamental information-geometrical meaning [6, 7, 39] since it is proportional to the Bures metric g_λ^{Bures}:

$$QFI(\rho_\lambda) = 4g_\lambda^{Bures}. \tag{5}$$

g_λ^{Bures} provides the infinitesimal geometric distance $ds^2(\rho_\lambda,\rho_{\lambda+\delta\lambda})$ between two neighboring quantum states and their statistical distinguishability; thus, it measures how well the states, and thus the parameter λ, can be discriminated.

We now show that $QFI(\rho_\lambda)$ can in general be connected to the variation of the coherence of the basis $\mathcal{B}_{\boldsymbol{\alpha}}^\lambda$ with respect to the state ρ_λ when the latter undergoes an infinitesimal change $\rho_\lambda \to \rho_{\lambda+\delta\lambda}$, with $\delta\lambda \ll 1$. In general, the coherence of given basis $\mathcal{B}_{\mathbf{x}}$ with respect to a state ρ can be measured by the relative entropy of coherence [41, 43]

$$Coh_{\mathcal{B}_{\mathbf{x}}}(\rho) = -\mathcal{V}(\rho) + H\left(p_x\right) \tag{6}$$

where $\mathcal{V}(\rho)$ is the von Neumann entropy of ρ and $H\left(p_x\right)$ is the Shannon entropy of the probability distribution $p_x = \langle x|\rho|x\rangle$. Now, it is well known that the QFI can be expressed as

$$QFI(\rho_\lambda) = \left[\partial_{\delta\lambda}^2 D(p_\alpha^{\lambda+\delta\lambda}|p_\alpha^\lambda)\right]_{\delta\lambda=0} \tag{7}$$

where $D(p_\alpha^{\lambda+\delta\lambda}|p_\alpha^\lambda)$ is the relative entropy between the probability distributions $p_\alpha^\lambda = \left\langle\alpha^\lambda|\rho_\lambda|\alpha^\lambda\right\rangle$ and $p_\alpha^{\lambda+\delta\lambda} = \left\langle\alpha^\lambda|\rho_{\lambda+\delta\lambda}|\alpha^\lambda\right\rangle$; the latter being the probabilities of the measurement defined by $\mathcal{B}_{\boldsymbol{\alpha}}^\lambda$ realized on ρ_λ and $\rho_{\lambda+\delta\lambda}$ respectively. The latter equation can be also written as

$$QFI(\rho_\lambda) = \left[-\partial_{\delta\lambda}^2 H(p_\alpha^{\lambda+\delta\lambda}) + \partial_{\delta\lambda}^2\mathcal{X}(p_\alpha^{\lambda+\delta\lambda}|p_\alpha^\lambda)\right]_{\delta\lambda=0} \tag{8}$$

where $\mathcal{X}(p_\alpha^{\lambda+\delta\lambda}|p_\alpha^\lambda)$ is the cross entropy of the two distributions. On the other hand, from (6) we obtain

$$-\left[\partial_{\delta\lambda}^2 Coh_{\mathcal{B}_{\boldsymbol{\alpha}}^\lambda}(\rho_{\lambda+\delta\lambda})\right]_{\delta\lambda=0} = -\left[\partial_{\delta\lambda}^2 H(p_\alpha^{\lambda+\delta\lambda})\right]_{\delta\lambda=0} + \left[\partial_{\delta\lambda}^2\mathcal{V}(\rho_{\lambda+\delta\lambda})\right]_{\delta\lambda=0} \tag{9}$$

By comparing (8) and (9) we obtain the following

Proposition 1. *For a general estimation processes, the QFI is related to the second order variation of the coherence of the Symmetric Logarithmic Derivative eigenbasis $\mathcal{B}^{\lambda}_{\boldsymbol{\alpha}}$ as:*

$$- \left[\partial^2_{\delta\lambda} Coh_{\mathcal{B}^{\lambda}_{\boldsymbol{\alpha}}}(\rho_{\lambda+\delta\lambda})\right]_{\delta\lambda=0} = f(\rho_\lambda) + QFI(\rho_\lambda) \tag{10}$$

with $f(\varrho_\lambda) = -\left[\partial^2_{\delta\lambda}\mathcal{X}(p^{\lambda+\delta\lambda}_{\alpha}|p^{\lambda}_{\alpha})\right]_{\delta\lambda=0} + \left[\partial^2_{\delta\lambda}\mathcal{V}(\rho_{\lambda+\delta\lambda})\right]_{\delta\lambda=0}$.

Proposition 1 is central in our analysis, as it establishes a link between the optimal precision in an estimation process, the geometry of quantum states and the coherence of the optimal measurement basis. The found connection is further strengthened whenever $f(\rho_\lambda) = 0$ and $Coh_{\mathcal{B}^{\lambda}_{\boldsymbol{\alpha}}}(\rho_{\lambda+\delta\lambda})$ has a critical point in $\delta\lambda = 0$, so that the QFI *can be expressed as the curvature of* $Coh_{\mathcal{B}^{\lambda}_{\boldsymbol{\alpha}}}(\rho_{\lambda+\delta\lambda})$ *around a maximum.* It turns out that these conditions are always verified for pure states. As proven in Appendix I for pure states $\rho_\lambda = |\psi_\lambda\rangle\langle\psi_\lambda|$ the following holds:

Proposition 2. *If ρ_λ is pure, then*

$$- \left[\partial^2_{\delta\lambda} Coh_{\mathcal{B}^{\lambda}_{\boldsymbol{\alpha}}}(\rho_{\lambda+\delta\lambda})\right]_{\delta\lambda=0} = 4g^{FS}_{\lambda} \tag{11}$$

i.e., the second order variation of $-Coh_{\mathcal{B}^{\lambda}_{\boldsymbol{\alpha}}}(\rho_{\lambda+\delta\lambda})$ *is proportional to the Fubini-Study metric* g^{FS}_{λ}*, the restriction of the Bures metric to the projective space* $\mathcal{PH}$ *of pure states.*

Therefore $-\left[\partial^2_{\delta\lambda} Coh_{\mathcal{B}^{\lambda}_{\boldsymbol{\alpha}}}(\rho_{\lambda+\delta\lambda})\right]_{\delta\lambda=0}$ determines the geometry of pure quantum states and the relative estimation bounds that can be derived within the Cramer-Rao formalism. For pure states L_λ has only two nonzero eigenvalues, corresponding to the eigenvectors $|\psi_\lambda\rangle$ and $\frac{d}{d\lambda}|\psi_\lambda\rangle$. Thus the estimation process in fact happens in the single-qubit space spanned by $|\psi_\lambda\rangle$ and $\frac{d}{d\lambda}|\psi_\lambda\rangle$, and what matters is the change of the coherence in this subspace. For mixed states, in general $f(\rho_\lambda) \neq 0$. However, as we will prove below by means of specific examples, in many cases of interest one has $f(\rho_\lambda) \ll QFI(\rho_\lambda)$ so that the relation $-\left[\partial^2_{\delta\lambda} Coh_{\mathcal{B}^{\lambda}_{\boldsymbol{\alpha}}}(\rho_{\lambda+\delta\lambda})\right]_{\delta\lambda=0} \approx QFI(\rho_\lambda) \approx 4g^{Bures}_{\lambda}$ approximately holds and the variation of coherence is the leading term that determines the geometry of mixed quantum states and the relative estimation bounds.

We finally observe that Eq. (10) and Eq. (11) are very general since they hold whatever the process that induce the infinitesimal change $\rho_\lambda \to \rho_{\lambda+\delta\lambda}$. In the following Section we specialize our analysis to unitary phase estimation processes in which $\rho_{\lambda+\delta\lambda} = U_{\delta\lambda}\rho_\lambda U^{\dagger}_{\delta\lambda}$, where $U_{\delta\lambda}$ is a unitary operator. In Section IV we extend the discussion to some relevant non-unitary evolutions.

III. COHERENCE IN PHASE ESTIMATION; PURE AND MIXED PROBE STATES

In this section we analyze the above results in the case of unitary Quantum Phase Estimation processes where

$$\rho_\lambda = \exp\left(-i\lambda G\right)\rho_0 \exp\left(i\lambda G\right), \tag{12}$$

G is a Hermitian traceless operator, and the unknown phase λ is the parameter to be estimated. In this case QFI is independent of λ, and it is sufficient to address the estimation problem for $\lambda = 0$ [5].

A. The single-qubit case

We start by analyzing the single-qubit case in which ρ_0, G and the generic measurement basis $\mathcal{B}$ can be defined in the Bloch sphere formalism in terms of the vectors $\vec{z}, \hat{\gamma}, \hat{b}$ respectively as follows. Without loss of generality we choose the single qubit state

$$\varrho_0 = (1 + \vec{z}\cdot\boldsymbol{\sigma})/2 \tag{13}$$

where $\vec{z} = z\hat{z} = z(0,0,1)$, $0 \le z \le 1$ and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli matrices; and the phase generator

$$G = \gamma\left(\hat{\gamma}\cdot\boldsymbol{\sigma}\right) \tag{14}$$

with $\hat{\gamma} = (\sin\delta, 0, \cos\delta)$, such that its eigenbasis lies in the $\hat{x}\hat{z}$ plane, forming an angle $0 \le \delta \le \frac{\pi}{2}$ with $\hat{z}$.
A generic measurement basis $\mathcal{B}_{\hat{b}}$ is defined by the projectors $\Pi^{\hat{b}}_{\pm} = (1 \pm \hat{b}\cdot\boldsymbol{\sigma})/2$ with $\hat{b} = \{\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta\}$. For a mixed state ($z < 1$) the FI in $\lambda = 0$ is given by (for the proof, see appendix II):

$$FI(\mathcal{B}_{\hat{b}}, \rho_0, G) = 4\frac{(\vec{\gamma}\times\vec{z}\cdot\hat{b})^2}{1-(\vec{z}\cdot\hat{b})^2}. \tag{15}$$

The maximisation of FI over the measurement basis has a unique solution ([6]) and leads to the choice of the eigenbasis $\mathcal{B}_{\hat{\alpha}}$ of the SLD, that in our case corresponds to choosing $\hat{b} = \hat{\alpha} = \{0, 1, 0\} \propto \hat{\gamma}\times\hat{z}$. As for the coherence of $\mathcal{B}_{\hat{\alpha}}$ one has that

$$[\partial_\lambda Coh_{\mathcal{B}_\alpha}(\rho_\lambda)]_{\lambda=0} = 0 \tag{16}$$

$$FI(\mathcal{B}_\alpha, \rho_0) = QFI(\rho_0, G) = -\left(\partial^2_\lambda Coh_{\mathcal{B}^0_\alpha}(\rho_\lambda)\right)_{\lambda=0} \tag{17}$$

and therefore one obtains result (10), with $f(\rho_\lambda) = 0$ (proof in Appendix II). Since $\mathcal{B}_{\hat{\alpha}}$ is unique one has that *for single qubit mixed states the necessary and sufficient condition for attaining the Cramer-Rao bound is the maximization of the coherence of the measurement basis with respect to the state* ρ_0*, and the QFI coincides with its second order variation.*
The optimization of $QFI(\rho_0, G)$ with respect to G leads to the choice of $\hat{\gamma}\cdot\hat{z} = 0$ that corresponds to the maximization of the coherence of the eigenbasis of G with respect to ρ_0, as has been highlighted in [47–49]. Taking into account both maximizations, one has $QFI = 2z^2Tr[G^2]$. Therefore our treatment allows to interpret the optimal estimation procedure as the one that takes advantage of the full strength of G in order to variate the relevant coherence i.e., that of the basis $\mathcal{B}_\alpha$.
For pure states the measurements axes $\hat{b}$ leading to QFI are not unique. However, $\hat{b} = \hat{\alpha}$ is the only choice that allows to attain the highest sensitivity in λ, and at the same time the lowest sensitivity with respect to small changes in the measurment angles $\delta\theta, \delta\phi$, possibly due to imperfections of the measurement apparatus, or to the impurity of the initial state (see Appendix II).

B. The N-dimensional case

We now pass to analyze the general case $N = \dim\mathcal{H} > 2$, where there is room for discussing the role of coherence vs correlations in estimation processes such as QPE. In order to do so we need to find a direct connection between QFI and the correlations relevant for the estimation. In this subsection we first describe the main points of our approach and give general formal results. We will then illustrate the results by means of specific examples in Sections III C, III D and III E.

The relation between coherence and the QFI given by (10) holds for any N and irrespectively of any *local* structure of the given Hilbert space $\mathcal{H}_N$. Correlations, instead, are typically defined between subsystems i.e., when a specific tensor factorization of the Hilbert space is chosen. An $N-$dimensional Hilbert space in general admits several (possibly inequivalent) factorizations in subsystems $\mathcal{H}_{n_i}$

$$\mathcal{H}_N = \otimes_i \mathcal{H}_i$$

with $\Pi_i n_i = N,\ n_i = dim\mathcal{H}_i$. Accordingly different definitions of bi- or multi-partite (quantum) correlations are possible. The different factorizations are called *tensor product structures* (TPS). Which decomposition is "relevant" or useful should in general be suggested by the problem at hand.

The first step of our approach is therefore to identify such relevant factorization. It turns out that, under some hypotheses, for QPE one can use the eigendecomposition of L_0 to uniquely identify a factorization of the Hilbert space into the product of a single qubit and an $N/2-$dimensional subspace

$$\mathcal{H}_N \sim \mathcal{H}_2\tilde{\otimes}\mathcal{H}_{N/2} \tag{18}$$

We will refer to (18) as the *reference TPS* , indicate it with TPS^R and, for sake of clarity, use for the corresponding tensor product operator the symbol $\tilde{\otimes}$ to distinguish it from other TPS (e.g., the standard TPS on M qubits). As we will show below, in TPS^R, the SLD can be written as $L_0 = O_2\tilde{\otimes}O_{N/2}$ where O_2, $O_{N/2}$ are operators acting locally on $\mathcal{H}_2$ and $\mathcal{H}_{N/2}$ respectively. Therefore the eigenvectors of O_2, $O_{N/2}$ form a product basis $\mathcal{B}_2\tilde{\otimes}\mathcal{B}_{N/2}$ and the relative projectors $\{\Pi_\pm \otimes \Pi_k\}$, $k = 1,..,N/2$ define: a global von Neumann experiment on $\mathcal{H}_N$, whose outcomes are distributed according a joint probability distribution $p^\lambda_{\pm,k} = Tr\left[\Pi_\pm\tilde{\otimes}\Pi_k\rho_\lambda\right]$; and local experiments with outcomes distributed according to the marginals $p^\lambda_\pm = Tr\left[\Pi_\pm\tilde{\otimes}\mathbb{I}_{N/2}\rho_\lambda\right]$ and $p^\lambda_k = Tr\left[\mathbb{I}_2\tilde{\otimes}\Pi_k\rho_\lambda\right]$.

The second step of our approach is based on the use of a relation between coherence and (classical) correlations that can be found by applying the definition of coherence (6) to the case of product bases [56]. In particular for $\mathcal{B}_2\tilde{\otimes}\mathcal{B}_{N/2}$ the coherence function can be written as :

$$Coh_{\mathcal{B}_{\pm,k}}(\rho_\lambda) \;=\; -\mathcal{V}(\rho_\lambda) + H(p^\lambda_\pm) + H(p^\lambda_k) - \mathcal{M}^\lambda_{L_0}\left(p^\lambda_{\pm,k}\right) \tag{19}$$

where

$$\mathcal{M}^\lambda_{L_0}\left(p^\lambda_{\pm,k}\right) = H\left(p^\lambda_\pm\right) + H\left(p^\lambda_k\right) - H\left(p^\lambda_{\pm,k}\right) \tag{20}$$

is the classical mutual information for the probability distribution $p^\lambda_{\pm,k}$. We notice that expression (19) was used in [56] where it was shown that the efficiency of a communication protocol such as remote state preparation requires the maximization of the correlations between some relevant observables i.e., the maximization of the relative mutual information. For certain kinds of two-qubit states, relation (19) expresses a general trade-off between correlations and coherence: when the former is maximized the latter is correspondingly minimized. For remote state preparation the relation is between static resources stored in the system state. Here we will see that in dynamical processes such as quantum estimation protocols the relation is between the changes of correlations and coherence.

Let us now proceed to derive the general results that allow to define the TPS^R and to state the relations between QFI, coherence and correlations that follows from (19). The TPS^R construction builds on the properties of the eigendecomposition of L_0 when some hypotheses on ρ_0, G, L_0 are satisfied:

Proposition 3. *Under the following hypotheses: i) N is even; ii) the initial diagonal state $\rho_0 = \sum_n p_n|n\rangle\langle n|$ is full rank; iii) $\langle n|G|m\rangle \in \mathbb{R}\ \forall n,m$ i.e., G has purely real matrix elements when expressed in the eigenbasis of ρ_0; iv) and $L_{\lambda=0}$ is full rank then:*

3.1 L_0 is diagonal in a basis

$$\mathcal{B}_{\boldsymbol{\alpha}=(\pm,k)} = \{|\alpha_{i,k}\rangle\}, \qquad i = \pm, k = 1,..,N/2 \tag{21}$$

with eigenvalues that are opposite in pairs

$$\alpha_{\pm,k} = \pm\alpha_{+,k} \in \mathbb{R}\setminus\{0\} \tag{22}$$

3.2 The Hilbert space can be decomposed as $\mathcal{H}_N = \mathcal{H}_2\tilde{\otimes}\mathcal{H}_{N/2}$; the eigenvectors of the SLD can be written as

$$|\alpha_{i,k}\rangle = |i\rangle\tilde{\otimes}|k\rangle, \qquad i = \pm, k = 1,..,N/2 \tag{23}$$

and the SLD in its diagonal form can be written as

$$L_0 = S_y\tilde{\otimes}\sum_{k=1,..,N/2}\alpha^k_+\Pi_k \tag{24}$$

where $S_y = \Pi_+ - \Pi_-$ is a Pauli matrix acting locally on the single qubit sector $\mathcal{H}_2$ and $O_{N/2} = \sum_{k=1,..,N/2}\alpha^k_+\Pi_k$ is an operator that depends on the eigevalues of L_0 and acts locally onsubsystem $\mathcal{H}_{N/2}$

The proof of 3.1) is given in the Appendix III while the proof of 3.2) is given in Appendix IV. The first result depends on the fact that under the stated hypotheses L_0 is a Hermitian anti-symmetric operator. Result 3.2) relies on the fact that a possible way to induce a TPS is based on the observables of the system [52, 53]. Indeed, suppose one has a set of sub-algebras of Hermitian operators $\mathcal{A}_2$, $\mathcal{A}_{N/2}$ satisfying the following conditions: $i)$ commutativity, i.e., $\left[\mathcal{A}_2, \mathcal{A}_{N/2}\right] = 0$; $ii)$ completeness, i.e. the product of the observables belonging to $\mathcal{A}_2, \mathcal{A}_{N/2}$ allows to generate the full set of Hermitian operators over $\mathcal{H}_N$. Then $\mathcal{A}_2 \vee \mathcal{A}_{N/2} \cong \mathcal{A}_N$ and one can induce a factorization $\mathcal{H}_N = \mathcal{H}_2\tilde{\otimes}\mathcal{H}_{N/2}$ such that: each $O_2 \in \mathcal{A}_2$, $O_{N/2} \in \mathcal{A}_{N/2}$ acts locally on $\mathcal{H}_2$ and $\mathcal{H}_{N/2}$ respectively; the composition (product) of operators can be written as $O_2O_{N/2}\left(\mathcal{H}_2\tilde{\otimes}\mathcal{H}_{N/2}\right) = O_{N/2}O_2\left(\mathcal{H}_2\tilde{\otimes}\mathcal{H}_{N/2}\right) = \left(O_2\mathcal{H}_2\right)\tilde{\otimes}\left(O_{N/2}\mathcal{H}_{N/2}\right)$. In Appendix IV we show how the algebras $\mathcal{A}_2, \mathcal{A}_{N/2}$ can be explicitly constructed by taking appropriate sums of projectors $\Pi_{\pm,k} = |\alpha_{i,k}\rangle\langle\alpha_{i,k}|$ onto the eigenbasis $\mathcal{B}_{\boldsymbol{\alpha}=(\pm,k)}$ given by result 3.1).

Having defined TPS^R, we can now enounce the main result of this section. The relation between QFI, coherence and $\mathcal{M}^\lambda_{L_0}$ can be stated in the following way :

Proposition 4. *Under the same hypotheses of Proposition 3 one has:*

4.1 The relation $QFI = \left[-\partial^2_\lambda Coh_{\mathcal{B}_{\pm,k}}(\rho_\lambda)\right]_{\lambda=0} + f(\rho_\lambda)$ *is attained in correspondence of a critical point of* $Coh_{\mathcal{B}_{\pm,k}}(\rho_\lambda)$ *i.e.,* $\left[\partial_{\delta\lambda} Coh_{\mathcal{B}_{\pm,k}}(\rho_\lambda)\right]_{\delta\lambda=0} = 0$;

4.2 $\mathcal{M}^{\lambda=0}_{L_0} = 0$ *and* $\left(\partial_\lambda \mathcal{M}^\lambda_{L_0}\right)_{\lambda=0} = 0$ *i.e., the observables* S_y *and* $O_{N/2}$ *are uncorrelated for* $\lambda = 0$*, and therefore* $\lambda = 0$ *is a minimum for* $\mathcal{M}^\lambda_{L_0}$ *and* $\left(\partial^2_\lambda \mathcal{M}^\lambda_{L_0}\right)_{\lambda=0} \geq 0$

4.3 the QFI can be written as

$$QFI = FI_2 + \left(\partial^2_\lambda \mathcal{M}^\lambda_{L_0}\right)_{\lambda=0} \tag{25}$$

where $FI_2 = \left[\sum_{i=\pm} \frac{(\partial_\lambda p_i)^2}{p_i}\right]_{\lambda=0}$

4.4 Given the single qubit reduced density matrix $\xi_\lambda = Tr_{\mathcal{H}_{N/2}}[\rho_\lambda]$ *one has*

$$FI_2 \leq QFI(\xi_\lambda) \leq QFI$$

The proof is given in the last part of Appendix IV. Proposition 4 allows to give a new interpretation of the QFI and has the following several different consequences.

Result 4.1) shows that the Quantum Cramer-Rao bound is achieved in correspondence of a critical point of the coherence of the eigenbasis of the SLD with respect to ρ_λ. This typically corresponds to a *maximum.* On the other hand, result 4.2) shows that the correlations between the relevant observables defined by $L_0 = S_y \tilde{\otimes} O_{N/2}$ are minimized. These results mirror the general trade-off between correlations and coherence mentioned above [56]. Here the variation of coherence is maximized in correspondence of a minimum of the correlations between the relevant observables. As we discuss below and in the following examples the minimization of correlations, and in particular their complete absence, has some relevant consequences for the representation of the estimation procedure and its efficiency.

As for result 4.3), Equation (25) shows that the QFI can be expressed in terms of two contributions. The first term $FI_2 = \left[\sum_i (\partial_\lambda p_i)^2 / p_i\right]_{\lambda=0}$ is the Fisher information of a single qubit. Indeed, since

$$p^\lambda_\pm = Tr_{\mathcal{H}_N}\left[\Pi_\pm \otimes \mathbb{I}_{N/2}\rho\right] = Tr_{\mathcal{H}_2}\left[\Pi_\pm \xi_\lambda\right]$$

FI_2 is the Fisher Information corresponding to the measurement of the local S_y onto the reduced density matrix $\xi_\lambda = Tr_{\mathcal{H}_{N/2}}[\rho_\lambda]$. The other term is given by the second order variation of the correlations $\mathcal{M}^\lambda_{L_0} = \mathcal{M}\left(p^\lambda_{\pm,k}\right)$ between the observables defined by the eigendecompostion of L_0 via TPS^R. We notice that on one hand the connection found is between the QFI and specific *classical correlations* rather then quantum ones. On the other hand, since the estimation process is a dynamical one, the connection involves a variation of those correlations with λ. Indeed, since $\mathcal{M}\left(p^{\lambda=0}_{\pm,k}\right) = 0$, the relevant correlations have a minimum in $\lambda = 0$, and the efficiency of the estimation protocol depends on the "acceleration" with which those correlations are changed by the unitary evolution that impresses the phase onto the state.

The last result 4.4) derives on one hand from the fact that in general the basis defined by S_y does not correspond to eigenbasis of the SLD for ξ_λ, therefore in general $FI_2 \leq QFI(\xi_\lambda)$; and on the other hand from the fact that an estimation process realized on a subsystem gives in general a lower precision than an estimation realized on the whole system, hence $QFI(\xi_\lambda) \leq QFI$. The relevance of result 4.4) stems from the fact that, if one is able to evaluate ξ_λ, then by working on a single qubit system, independently on the dimension N, one can easily found a lower bound on the QFI i.e., $QFI(\xi_\lambda) \leq QFI$. This becomes quite relevant whenever one is interested in evaluating the scaling behavior of QFI with N. We will see an example of how property 4.4) can be fruitfully used in Section IV B, where we show that for certain noisy estimation processes based on GHZ-sates, ξ_λ can be evaluated and the scaling behavior of QFI can be deduced by means of $QFI(\xi_\lambda)$.

We close this subsection with a few comments. Overall, the decomposition given by (25), allows to unambiguously express the QFI in terms of a single qubit Fisher Information and classical correlations. This result is particularly interesting whenever $\left(\partial^2_\lambda \mathcal{M}^\lambda_{L_0}\right)_{\lambda=0} = 0$, so that $QFI = FI_2 = QFI(\xi_\lambda)$. In such cases the TPS^R construction shows that the estimation process is effectively a single-qubit one, whatever the dimension N of the original Hilbert space; and no (quantum) correlations in the probe state are involved in the process. This turns out to be the case in some relevant examples we discuss in the next section.

As a final remark, let us discuss the generality of the results obtained in Proposition 3 and 4. We notice that although seemingly restrictive, the hypotheses stated in Proposition 3 are actually quite general. Indeed, on one hand N even includes all multi-qubit states. On the other hand, as it can be seen from the following general expression for

QFI ([5])

$$QFI = 2\sum_{n\neq m} \frac{(p_n - p_m)^2}{(p_n + p_m)} |\langle n|G|m\rangle|^2$$

where $\{p_n, |n\rangle\}$ are the eigenvalues and eigenvectors of ρ_0, for each Hemitian operator $\tilde{G} \in \mathbb{C}^N \times \mathbb{C}^N$ one can always find a corresponding G such that $\langle n|G|m\rangle = |\langle n|\tilde{G}|m\rangle| \in \mathbb{R}\ \forall n, m$, and $Tr\left[\tilde{G}^2\right] = Tr\left[G^2\right]$, and such that both operators have the same QFI. Therefore, as for the estimation process, the choice of $\tilde{G}$ or G is equivalent and by using G our results can be applied. As for the requirement that L_0 is full rank, it can be easily relaxed as it will be shown in the specific examples below.

C. Examples: states with maximal QFI

We first focus on the situation in which given ρ_0 one seeks the operator with fixed norm γ,

$$G \in \mathcal{O}_\gamma = \left\{G \mid Tr[G^2] = 2\gamma^2\right\}$$

that allows to obtain the best estimation precision i.e., $QFI_{Max} = \max_{G\in\mathcal{O}_\gamma}(\rho_0, G)$. We now state the main results, while the detailed calculations are reported in Appendix V. Suppose $\rho_0 = \sum_{n=1}^{N} p_n|n\rangle\langle n|$ such that its eigenvalues are arranged in decreasing order, then the optimal operator reads

$$\bar{G} = G_{1N}\left(|1\rangle\langle N| + |N\rangle\langle 1|\right)$$

and

$$QFI_{Max} = 4\gamma^2 \frac{(p_1 - p_N)^2}{(p_1 + p_N)}$$

In this case L_0 is not full-rank, so the basis $\mathcal{B}_{\alpha=\pm,k}$ and the relative TPS^R are no longer unique. However, for all choices of bases $\mathcal{B}_{\alpha=\pm,k}$, Eq. (10) holds with $f(\rho_{\lambda=0}) = 0$, and the maximal QFI can be expressed in terms of the variation of the coherence of any of such bases as:

$$QFI_{Max} \;=\; -\left[\partial^2_\lambda Coh_{\mathcal{B}_\alpha}(\rho_\lambda)\right]_{\lambda=0} = 4g^{Bures}_\lambda$$

As for the decomposition defined in Proposition 4, QFI_{Max} can be expressed in terms of two contributions whose values, due to the non-uniqueness of $\mathcal{B}_{\alpha=\pm,k}$ depend on the specific basis chosen. For all choices of basis, one has that

$$\begin{aligned} FI_2 \;&\geq\; QFI_{Max}\cdot(p_1 + p_N) \\ \left(\partial^2_\lambda \mathcal{M}^\lambda_{L_0}\right)_{\lambda=0} \;&\leq\; QFI_{Max}\cdot(1 - p_1 - p_N) \end{aligned}$$

In general, when the initial state is mixed, the action of $\bar{G}$ which is relevant for the estimation is partially to change a single qubit coherence, as measured by FI_2, and partially to change the relevant correlations such that $\left(\partial^2_\lambda \mathcal{M}^\lambda_{L_0}\right)_{\lambda=0} \neq 0$. In the pure state limit $p_1 \to 1, p_N \to 0$, the contribution of the correlations drops to zero, $FI_2 \to QFI$ and one recovers the result (11). For pure states the estimation process is therefore in essence a single-qubit one, where the action of $\bar{G}$ that is relevant for the estimation is only focused to change a single qubit coherence, while the correlations disappear from the picture; and the $QFI_{Max} = 4\gamma^2$ is the maximal QFI one can obtain at fixed N and $Tr\left[G^2\right]$.

D. Examples: class of separable states

We now introduce a family of states that allow us to discuss different aspects of estimation protocols. On one hand we identify coherence as the relevant resource for the estimation and we discuss the possible role of different kinds of quantum correlations such as entanglement and discord [54]. On the other hand the following analysis will also give us the opportunity of studying estimation protocols based on mixtures of GHZ and NOON states and to lay the foundation for our later discussion about noisy estimation processes based on GHZ probes (section IV B).

The states we focus on can be written as

$$\rho_0 = \sum_{k=1}^{N/2} p_k \tau_k \otimes \Pi_k \tag{26}$$

defined by $\{\tau_k, p_k\}_{k=1}^{N/2}$, where

$$\tau_k = (\mathbb{I} + \vec{n}_k \cdot \vec{\sigma})/2, \qquad \vec{n}_k = h_k(\cos\phi_k, \sin\phi_k, 0)$$

are single qubit states, with $\vec{n}_k = h_k(\cos\phi_k, \sin\phi_k, 0)$ the relative Bloch vector lying in the xy plane and $|\vec{n}_k| = h_k \leq 1\ \forall k$.

We now choose as shift operator $G = \sigma_z \otimes \mathbb{I}_{N/2}$. In order to identify the resource that is relevant for the estimation of λ we can work in the original TPS in which the state is defined and later discuss the results in TPS^R (details of the calculations can be found in Appendix V. Since the states are block diagonal, the SLD simply reads $L = \oplus_k L_k$ [63] with single-qubit contributions given by $L_k = 2i\eta\hat{\alpha}_k \cdot \bar{\sigma}$ and $\hat{\alpha}_k = \hat{n}_k \times \hat{z}$. One has that for such states $f(\rho_\lambda) = 0$ and

$$\begin{aligned} QFI &= \sum_k p_k QFI_k \\ QFI_k &= -\partial_\lambda^2 Coh_{\mathcal{B}_k}(\tau_k) = 4h_k^2 \end{aligned} \tag{27}$$

i.e., the QFI *is the weighted sum of the second order variations of single qubit coherences.* Therefore the estimation process is in essence a single qubit one and *it can be seen as an estimation process carried over in parallel on* $N/2$ *qubits.*

In general $\vec{n}_k \neq \vec{n}_h,\ h \neq k$, the state ρ_0, with respect to the original TPS has zero bipartite entanglement but non-zero discord. However, the essential resource is coherence rather than discord (as also suggested in [55]). Indeed, if one fixes the value of QFI, the latter can be achieved by a whole class of states defined by $\{\tau_k, p_k\}_{k=1}^{N/2}$, with different N, different purities, very different amount of quantum correlations (as measured by the discord) between the single qubit and the $N/2-$dimensional system; and most importantly *irrespectively of the latter.* Moreover, suppose one compares two states ρ_1 and ρ_2 such that they differ only for the direction of the Block vector $\vec{n}_h$ pertaining to a given τ_h. Suppose for example that in ρ_2, $\vec{n}_h$ does not lye in the xy plane, then $\vec{n}_h \cdot \hat{z} \neq 0$ and $QFI(\rho_2) \leq QFI(\rho_1)$ (see Appendix V): the presence of the kind of discord implied by this choice of $\vec{n}_h$ in ρ_2 would therefore be *detrimental* for the estimation process.

As for the interpretation of the result in the reference TPS^R we restrict to the simple case in which all single qubit states τ_k are pure. Given the eigenvectors $|\alpha_{\pm,k}\rangle$ corresponding to each L_k one can define

$$|\alpha_{\pm,k}\rangle \;=\; |\pm\rangle\tilde{\otimes}|k\rangle.$$

and one has that in general $\tilde{\otimes} \neq \otimes$ i.e., the TPS^R induced by L_0 is in general different from the original TPS. In order to derive the decomposition (25) it is sufficient to evaluate $p^\lambda_{\pm,k} = \langle\alpha_{\pm,k}|\rho_\lambda|\alpha_{\pm,k}\rangle = \langle\pm|\tilde{\otimes}\langle k|\rho_\lambda|\pm\rangle\tilde{\otimes}|k\rangle$; then one can easily find (Appendix V) that for this general class of states $\left(\partial^2_\lambda \mathcal{M}^\lambda_{L_0}\right)_{\lambda=0} = 0$ so that $FI_2 = QFI$ and no correlations are involved in the estimation process.

In the following we will focus on two subclasses of states of the type (26) that have been proposed for use in phase estimation setups.

E. Examples: GHZ states

Pure GHZ states are a prototypical case often presented in the literature [24, 25], in which the precision in the estimation of λ is shown to have an Heisenberg scaling [24], and in which the role of entanglement has been often discussed. GHZ-like states are of the kind

$$|GHZ_k^\pm\rangle = \left(|k\rangle_M \pm |\bar{k}\rangle_M\right)/\sqrt{2} \tag{28}$$

where $|k\rangle_M \equiv |k_M, .., k_1\rangle,\ k_i \in \{0,1\}$ is an M-qubit state, $k = \sum_{i=1}^M 2^{(i-1)}k_i$, $k = 0, .., 2^{M-1}-1$ and $|\bar{k}\rangle_M = |\bar{k}_M, .., \bar{k}_1\rangle$ represents the binary logical negation of k. The set $\{|GHZ_k^\pm\rangle\}$ forms the GHZ-basis for $\mathcal{H}_{2^M}$. In order to apply our framework we discuss the following general mixed state

$$\sum_{k=0}^{N/2-1} p_k |GHZ_k^+\rangle\langle GHZ_k^+| \tag{29}$$

The state is a mixture of the GHZ basis states $|GHZ_k^+\rangle$ and for $p_0 = 1$ one has the typical example of pure GHZ state $|GHZ_0^+\rangle = (|00...0\rangle + |11...11\rangle)/\sqrt{2}$ discussed in the literature. As shift operator one typically chooses $U_\lambda = \exp -i\lambda G$ with $G = \sum_{h=1}^{M} \sigma_z^h$. The above class of states allows us to develop our treatment of the estimation problem directly in the TPS^R representation, to relate the QFI to the relevant coherence variations and to analyze its decomposition according to (25).

We start our analysis by noticing that $G|GHZ_k^+\rangle = (M - 2|k|)\,|GHZ_k^-\rangle$, thus the evolution does not couple the various sectors k. Here $|k|$ is the number of ones in the binary representation of k and $(M - 2|k|)$ is the difference between the number of zeros and ones; since $k = 0, .., 2^{M-1} - 1$, then $|k| \in \{0, .., M - 1\}$. Correspondingly the SLD has a block diagonal form $L = \oplus_k L_k$ with

$$L_k = 2ip_k(M - 2|k|)\left(|GHZ_k^+\rangle\langle GHZ_k^-| - |GHZ_k^-\rangle\langle GHZ_k^+|\right). \tag{30}$$

In each block the eigenvectors are $|\alpha_{\pm,k}\rangle = \left(|GHZ_k^+\rangle \pm i|GHZ_k^-\rangle\right)/\sqrt{2}$. By writing $|\alpha_{\pm,k}\rangle \doteq |\pm\rangle\tilde{\otimes}|k\rangle$ we define the TPS^R and the whole Hilbert space can be written as $\mathcal{H}_2 \otimes \mathcal{H}_{N/2}$ with $N/2 = 2^{M-1}$.Each of the GHZ basis states appearing in (29) can be written as:

$$|GHZ_k^+\rangle = \frac{(|+\rangle + |-\rangle)}{\sqrt{2}}\tilde{\otimes}|k\rangle = |0\rangle_z\tilde{\otimes}|k\rangle \tag{31}$$

In this TPS^R one has that:

$$\begin{aligned} G \quad &= \quad S_x\tilde{\otimes}\sum_k (M - 2|k|)\,\Pi_k \\ L \ &= 2S_y\tilde{\otimes} \ \sum_k (M - 2|k|)\,\Pi_k \end{aligned}$$

where S_x, S_y, S_z are the Pauli operators acting on $\mathcal{H}_2$ (see Appendix V for a derivation). The initial state reads

$$\rho_0 \ = \ |0\rangle_{zz}\langle 0|\tilde{\otimes}\sum_k p_k \Pi_k, \tag{32}$$

(where $S_z|0\rangle_z = |0\rangle_z$), it is a product state in the TPS^R, it is in general mixed and it is therefore a special kind of the states (26).

The evolved state is

$$\rho_\lambda \ = \sum_{k=0}^{N/2-1} \ p_k \left\{\exp\left[-i\lambda\,(M - 2|k|)\ \ S_x\right]|0\rangle_{zz}\langle 0|\exp\left[-i\lambda\,(M - 2|k|)\ \ S_x\right]\right\}\tilde{\otimes}\Pi_k.$$

We start by analyzing the pure state case $p_0 \to 1$, i.e., $|GHZ_0^+\rangle = (|00...0\rangle + |11...11\rangle)/\sqrt{2}$. By tracing out the $\mathcal{H}_{N/2}$ system

$$Tr_{\mathcal{H}_{N/2}}\left[U_\lambda|\psi_0\rangle\langle\psi_0|U_\lambda^\dagger\right] = \exp\left(-i\lambda M S_x\right)|0\rangle_{zz}\langle 0|\exp\left(i\lambda M S_x\right). \tag{33}$$

one can clearly see that estimation process based on the multi-qubit $|GHZ_0^+\rangle$ is completely equivalent to a single qubit multi-round strategy [30] where a single qubit operator $\exp\left(-i\lambda\sigma_x\right)$ is applied M times to the initial state $|0\rangle_z$. In both cases the link with coherence can be found within our approach and

$$QFI = \left[-\partial_\lambda^2 Coh_{\mathcal{B}_\alpha}(|0^\lambda\rangle_{zz}\langle 0^\lambda|)\right]_{\lambda=0} = 4M^2 \tag{34}$$

where $|0^\lambda\rangle_z = \exp\left(-i\lambda M S_x\right)|0\rangle_z$. The Heinsenberg scaling with M therefore has the very same root both in the single qubit multi-round protocols and multi-qubit GHZ pure state case: *it is the ability of G to vigorously change the relevant single-qubit coherence that allows for such scaling.*

As for the decomposition (25) we notice that the evolution takes place in the $k = 0$ sector thus no correlations between subsystem $\mathcal{H}_2$ and $\mathcal{H}_{N/2}$ are created and one has that $QFI = FI_2$ and $\left(\partial_\lambda^2 \mathcal{M}_{L_0}^\lambda\right)_{\lambda=0} = 0$. Given the previous discussion one may therefore argue that correlations, and in particular entanglement, play no role in the estimation process. As shown by this simple example, by selecting the proper TPS^R one can formally establish an equivalence between two seemingly different estimation procedures and find the common resource that is at the basis of their efficiency.

We now pass to analyze the general mixed state (29). Within the given TPS^R representation it is manifest that during the whole evolution the operator U_λ does correlate subsystems $\mathcal{H}_2$ and $\mathcal{H}_{N/2}$, but it never creates entanglement. Since $L = \oplus L_k$ one again obtains

$$QFI = \sum_k p_k QFI_k = 4 \sum_k \left(M - 2|k|\right)^2 p_k = 4\Delta^2 G \tag{35}$$

where $\Delta^2 G$ is the variance of G. As for the relation with coherence, for each sector k one has

$$QFI_k \;=\; \left[-\partial^2_\lambda Coh_{\mathcal{B}_\alpha}(\tau_k^\lambda)\right]_{\lambda=0} = 4\left(M - 2|k|\right)^2$$

where $\tau_k^\lambda = \exp\left(-i\lambda\left(M - 2|k|\right)S_x\right)|0\rangle_{zz}\langle 0|\exp\left(i\lambda\left(M - 2|k|\right)S_x\right)$. Therefore, in each sector the QFI_k is given by the second order variation of a single qubit coherence, and the estimation process can be seen as a parallelized version of a single qubit multi-round one, where in each of the 2^{M-1} sectors labeled by k the single qubit $|0\rangle_z$ is rotated in the zy plane by $\exp\left(-i\lambda(M - 2|k|)\,S_x\right)$. The global QFI is therefore an average of the sectors' single qubit QFI_k's or equivalently of sectors' single qubit second order coherences variations.

We now discuss the decomposition of the QFI according to Proposition 4. Here L_0 is full rank and one can uniquely write (see Appendix V)

$$FI_2 = \left(M - 2\sum_k |k| p_k\right)^2 \tag{36}$$

such that in general $\left(\partial^2_\lambda \mathcal{M}^\lambda_{L_0}\right)_{\lambda=0} = QFI - FI_2 \geq 0$. In order to give a physical interpretation of the various terms composing the QFI we define the following operators:

$$O_a = S_a \otimes \sum_k \left(M - 2|k|\right)\Pi_k \tag{37}$$

with $S_a,\ a = x, y, z$ Pauli matrices on $\mathcal{H}_2$, and $Tr\left[O_a O_b^\dagger\right] = 0$ when $a \neq b$. We have that $O_x = G$, $2O_y = L_0$ and it holds $QFI = 4\Delta^2 O_x = 4\Delta^2 O_y$. As for the operator O_z, the latter is diagonal in the GHZ-basis and therefore it commutes with ρ_0. Furthermore O_z has the same set of eigenvalues of $O_x = G$ i.e, $\{M - 2|k|\}$. Since $k = 0, .., N/2 - 1 = 2^{M-1} - 1$ we now can interpret the $\mathcal{H}_{2^{M-1}} = span\{|k\rangle\}$ sector of the of the TPS^R as an $M-1$ qubit system, such that $|k\rangle = |k_{M-1}, .., k_1\rangle$ where $(k_{M-1}, .., k_1)$ is the $(M-1)$-digits binary representation of k. In this way the operator $\sum_k\left(M - 2|k|\right)\Pi_k$ can be written as

$$\sum_k \left(M - 2|k|\right)\Pi_k \;=\; \mathbb{I}_{2^{M-1}} + \hat{S}_z$$

where $\hat{S}_z = \sum_{t=0}^{M-1}\sigma_z^t$ is the total angular momentum along the z direction for an $(M-1)$ qubits system. $\hat{S}_z$ is diagonal in the $|k\rangle$ basis and its eigenvalues are $\{M - 1 - 2|k|\}_{k=0}^{M-1}$. With this representation the shift operator can written as

$$G \;=\; S_x \otimes \mathbb{I}_{2^{M-1}} + S_x \otimes \hat{S}_z \tag{38}$$

while

$$O_z \;=\; S_z \otimes \mathbb{I}_{2^{M-1}} + S_z \otimes \hat{S}_z \tag{39}$$

The above picture based on TPS^R allows to see that the dynamical evolution enacted by G is fully equivalent to as "system-bath" interaction where: the role of the system is played by the single qubit, its initial state being $|0\rangle_z$; the role of the bath by the $(M-1)$-qubits system defined above, its initial state being $\sum_k p_k \Pi_k$; and G as described in (38) can be seen as a system-bath interaction Hamiltonian. If one is able to prepare the state (32) and to realize the interaction Hamiltonian (38) the equivalence given by the description in the TPS^R provides *an alternative way* *of enacting the estimation procedure that is based on the preparation of (mixtures) of M qubits product states and does not require the preparation of initial states that are highly entangled such as the GHZ ones.* While the kind of interaction described by (38) may be difficult to realize in a M-qubit system, the above picture suggest that in order to implement the described estimation process one could resort to an analogous interaction between a single qubit and a spin-j systems prepared in an eigenstate of the corresponding $\hat{S}_z$.

As for the decomposition of QFI, it turns out that the relevant quantities can be simply expressed in terms of O_z as

$$\begin{aligned} FI_2 &= 4\langle O_z\rangle^2 \\ \left(\partial^2_\lambda \mathcal{M}^\lambda_{L_0}\right)_{\lambda=0} &= 4\Delta^2 O_z \end{aligned}$$

and by using (39) one has

$$\begin{aligned} \langle O_z\rangle &= 1 + \left\langle S_z \otimes \hat{S}_z \right\rangle \\ \Delta^2 O_z &= \Delta^2 \left(S_z \otimes \hat{S}_z\right) \end{aligned}$$

where we have used the fact that for the state (32) $\left\langle \left(\mathbb{I}_2 \otimes \hat{S}_z\right)\right\rangle = \left\langle S_z \otimes \hat{S}_z\right\rangle$.Therefore the value of FI_2 is determined by the average of interaction operator $S_z \otimes \hat{S}_z$ while $\left(\partial^2_\lambda \mathcal{M}^\lambda_{L_0}\right)_{\lambda=0}$ is determined by its variance thus providing a physical interpretation of these quantities. In the above picture, the effect of the system-bath interaction is to correlate the two subsystems $\mathcal{H}_2$ and $\mathcal{H}_{2^{M-1}}$ and the effect on the single qubit is a conditional rotation around the $\hat{x}$ axis that depends on $(M-2|k|)$. Furthermore, when $p_0 = 1$ the initial state is pure and it can be represented as $|\psi_0\rangle = |GHZ_0^+\rangle$ or in the TPS^R as $|\psi_0\rangle = |0\rangle_z \tilde{\otimes} |0\rangle$. In this case the bath does not evolve under the action of G and the two subsystems *remains uncorrelated during the evolution*, such that in particular $\left(\partial^2_\lambda \mathcal{M}^\lambda_{L_0}\right)_{\lambda=0} = \Delta^2 \left(S_z \otimes \hat{S}_z\right) = 0$. The evolution can now be represented in the single qubit sector only as (33) such that

$$QFI = FI_2 = 4\langle O_z\rangle^2$$

and the QFI can be alternatively interpreted as: the square of the average value of O_z; the variance of the system-bath interaction Hamiltonian $O_x = G$; or the second order variation of a single qubit coherence. Therefore, once the evolution has taken place, as for the estimation process *one only needs to realize the single qubit measurement* defined by the L_0 eigenbasis $\{|0\rangle_y, |1\rangle_y\}$, and the estimation precision is again provided by the second order variation of a single qubit coherence.

We conclude this section by discussing the possibility of achieving a quasi-Heisenberg scaling of QFI with GHZ mixed states of the kind (29) and large number of qubits M or alternatively, thanks to the TPS^R representation, with a single qubit system coupled to a $M-1$ qubit one. We first notice that the scaling can be achieved whenever the distribution $\{p_k\}$ is mostly concentrated in the sectors k such that $|k| \ll M$ and/or $|k| \approx M$. This happens despite the fact that the initial state is mixed and, provided it satisfies the previous conditions whatever the distribution $\{p_k\}$ i.e., whatever the (quantum) correlations that may be built by the operator U_λ between the subsystems $\mathcal{H}_2$ and $\mathcal{H}_{N/2}$. This is exactly the kind of situation we will encounter when in Section IV where we discuss noisy estimation schemes base on GHZ states.

F. Examples: (mixed) NOON states

Another class of states of the type (26) that are worth mentioning are the following general (mixed) NOON states

$$\rho = p_0|0,0\rangle\langle 0,0| + \sum_{k=1}^{\infty} p_k \left\{[|k,0\rangle\langle k,0| + |0,k\rangle\langle 0,k|] + [\eta_k|k,0\rangle\langle 0,k| + \eta_k^*|0,k\rangle\langle k,0|]\right\} \tag{40}$$

where $|k,0\rangle = |k\rangle_a|0\rangle_b, |0,k\rangle = |0\rangle_a|k\rangle_b$ are Fock states of a two-modes (a,b) quantum optical system with k-photons in each mode respectively and $|0,0\rangle$ is the vacuum contribution. These states have been proposed for use in quantum phase estimation and have been thoroughly analyzed in Ref. [57]. Within each sector k the state can be represented as $\tau_k \otimes \Pi_k$ with

$$\tau_k = \begin{pmatrix} 1/2 & \eta_k \\ \eta_k^* & 1/2 \end{pmatrix} \tag{41}$$

i.e. as a single qubit state lying in the xy plane with Bloch vector $\bar{\eta}_k = |\eta_k| (\cos\arg\eta_k, \sin\arg\eta_k, 0)$; where $|\eta_k|$ determines the purity of state [58]. When now one applies a phase shift by means, for example, of the single mode operator $U(\lambda) = e^{-i\lambda a^\dagger a}$, one has

$$\eta_k|k,0\rangle\langle 0,k| \to e^{-ik\lambda}\eta_k|k,0\rangle\langle 0,k| \qquad \eta_k^*|k,0\rangle\langle 0,k| \to e^{ik\lambda}\eta_k^*|k,0\rangle\langle 0,k|$$

such that, aside for the vacuum contribution, the evolved state is analogous to (26) and reads

$$\rho_\lambda = p_0|0,0\rangle\langle 0,0| + \sum_{k=1}^{\infty} p_k \begin{pmatrix} 1/2 & e^{-ik\lambda}\eta_k \\ e^{ik\lambda}\eta_k^* & 1/2 \end{pmatrix} \otimes |k\rangle\langle k| \tag{42}$$

Again the process can be seen as a single qubit multi-round like one. Indeed, in each sector k the phase shift amounts to a rotation of the intial Bloch vector $\bar{\eta}_k$ of an angle $k\lambda$ in the xy plane. In general the estimation process is realized by enacting measurements represented by operators of the kind $A_M = |0,M\rangle\langle M,0| + |M,0\rangle\langle 0,M|$, which can be realized by means of interference measurements in a Mach-Zender interferometer set up. In our picture A_M corresponds to measuring the operator $S_x \otimes |M\rangle\langle M|$. In this set up the measurement is supposed to be fixed; in terms of the single qubit system, measuring A_M corresponds to measuring along the $\hat{x}$ axis. This measurement is optimal only when for the state τ_M^λ it projects onto the eigenstates of L_λ^M i.e., the SLD pertaining to the sub-block M. But this happens only for specific values of the phase λ i.e., when the condition $(\arg\eta_k - k\lambda) \mod \pi/2 \approx 0$ is verified with sufficiently good approximation[57], and the Bloch vector of the rotated state τ_M^λ is sufficiently close to the $\widehat{y}$ axis. The efficiency of the protocol depends on $QFI = p_M QFI_M$ where QFI_M is the usual single qubit QFI that corresponds to the second order variation of the coherence of the L_λ^M eigenbasis and p_M is the probability of projecting the state onto the sector M. In order to obtain a super-SQL scaling on one hand $p_M |\bar{\eta}_k|^2$ must be sufficiently greater than $1/M$; this in particular is true for a pure NOON state. On the other hand the condition $(\arg\eta_k - k\lambda) \mod \pi/2 \approx 0$ should be satisfied with sufficient accuracy.

IV. COHERENCE IN PHASE ESTIMATION IN PRESENCE OF NOISE

We now pass to analyze the situation in which the estimation protocols are affected by noise. In general the estimation processes may be "noisy" for different reasons: the imperfections of the state preparation procedure or the coupling of the probe state to the surrounding environment. Since in experiments typically one or both of these situations actually occur, noisy estimation processes have been and are being object of intense study[24, 64–72, 75]. In the following, we extend the framework laid down in previous sections to some specific cases of noisy evolution. This will allow us on one hand to describe the connection between phase estimation and coherence in presence of noise, and on the other hand to exemplify our approach in specific relevant examples.

A. Noise in state preparation and "commuting noise"

The extension of our approach is straightforward in at least two cases. The first case is when the noise acts *before* the phase encoding starts, i.e., the shift operator is applied to a mixed state ρ_0 that is the result of a noisy process or an imperfect state preparation procedure. Examples of this situation may well be the mixed states such as mixed GHZ and NOON states, analyzed in the previous section. In this case one can straightforwardly apply the results of Sec. II and III for mixed states, which hold for any dimension N: in particular, Eqs. (10) and (25) hold without change. A second case where our approach applies is when the system is coupled to a noisy environment but the map describing the overall process $\Lambda_{\lambda,\gamma}[\rho_0]$ is given by the composition of two different *commuting* maps $\Lambda_\lambda, \Lambda_\gamma$, enacting the coherent (Λ_λ) and decoherent (Λ_γ) part of the evolution (here γ is a generic parameter characterizing the decoherent process). Then $\Lambda_{\lambda,\gamma}[\rho_0] = \Lambda_\gamma[\Lambda_\lambda[\rho_0]] = \Lambda_\lambda[\Lambda_\gamma[\rho_0]]$ and the estimation process is equivalent to applying the coherent shift λ to the decohered state $\rho_\gamma = \Lambda_\gamma[\rho_0]$. This scenario includes several nontrivial cases of noisy evolutions (neat examples are given in [65, 68, 75]). It also includes the situation in which the unitary evolution is encoded in a decoherence free subsystem[73, 74], i.e., in which the unitary evolution enacting the phase shift Λ_λ takes places in a subspace that is left invariant by the noisy map Λ_γ. In this case, if the initial pure state ρ_0^{enc} belongs to the decoherence-free subsystem, it is left untouched by the noise ($\rho_0^{enc} = \Lambda_\gamma[\rho_0^{enc}]$) and the estimation process can take place by means of a proper encoding Λ_λ^{enc} of the phase shift. The latter corresponds to a unitary process and our arguments on the role of coherence can be easily applied

B. "Non-commuting" noise

The above examples show that the approach developed in previous sections can be straightforwardly extended to whole classes of noisy evolutions. But what happens when the noisy and the coherent map do not commute? In such general case the picture changes, mainly because the eigenvalues $\{\epsilon_i^\lambda\}$ of the output state of the evolution $\rho_\lambda = \Lambda_{\lambda,\gamma}(\rho_0)$ possibly depend on λ, and therefore $\mathcal{V}(\rho_\lambda)$ is not conserved. Then formula (10) holds with

$$f(\rho_\lambda) = f^{\chi}(\rho_\lambda) + f^{\epsilon}(\epsilon_i^\lambda) - QFI^c \tag{43}$$

where $f^{\chi}(\rho_\lambda) = -\left[\partial^2_{\delta\lambda}\mathcal{X}(p_\alpha^{\lambda+\delta\lambda}|p_\alpha^\lambda)\right]_{\delta\lambda=0}$is the usual term appearing in the noise-free case while the additional terms

$$\begin{aligned} f^{\epsilon}(\epsilon_i^\lambda) &= -\sum_i \left(\partial^2_{\delta\lambda}\epsilon_i^{\lambda+\delta\lambda}\right)_{\delta\lambda=0} \log \epsilon_i^\lambda \\ QFI^c &= \sum_i \left(\partial^2_{\delta\lambda}\epsilon_i^{\lambda+\delta\lambda}\right)^2_{\delta\lambda=0} / \epsilon_i^\lambda \end{aligned}$$

come from the variation of the eigenvalues of ρ_λ with λ. The term QFI^c is the "classical contribution" to the QFI due to the first order variation of the ϵ_i^λ with λ[6]. Since $QFI = QFI^c + QFI^Q$ where QFI^Q is the part depending on the variation of the eigenvectors of ρ_λ with λ, relation (10) can be further simplified as

$$-\left(\partial^2_{\delta\lambda}Coh\right)_{\delta\lambda=0} = f^{\epsilon}(\epsilon_i^\lambda) + f^{\chi}(\rho_\lambda) + QFI^Q \tag{44}$$

Discussing the previous relation in the most general case and drawing general conclusions on complex decoherent processes is a compelling but rather hard task. In the following we focus on a relevant example of noisy estimation processes that has been diffusely explored in the literature: phase estimation based on the M-qubit GHZ state in the presence of noise ([71]).

The relevance of this case stems on one hand from the fact that, as previously discussed, GHZ states in the noise-free case allow to attain the Heisenberg limit $QFI \propto M^2$. On the other hand, it is has been shown [66] that GHZ states allow to overcome the Standard Quantum Limit (SQL) $QFI \propto M$ even in the presence of noise (for specific kinds of noisy maps). In the following we show how our approach based on coherence applies and how the results seen in Section III E can be extended.

The specific setting we describe in the following has been put forward in [66] and we now briefly review it. Assume that the M-qubits systems undergoes a coherent phase shift and is subject to Pauli diagonal noise. Then its evolution is governed by the Markovian master equation

$$\partial_t\rho = \mathcal{H}(\rho) + \mathcal{L}(\rho) \tag{45}$$

where the unitary part enacting the phase shift is the same used for the pure state case and it is given by

$$\mathcal{H}(\rho) = -\frac{i\omega}{2}[\sum_h \sigma_z^h, \rho]. \tag{46}$$

while the decoherent part is given by

$$\mathcal{L}(\rho) = -\frac{\gamma}{2}\sum_h \left[\rho - \alpha_x\sigma_x^h\rho\sigma_x^h - \alpha_y\sigma_y^h\rho\sigma_y^h - \alpha_z\sigma_z^h\rho\sigma_z^h\right] \tag{47}$$

where $\sigma_a^h,\ a = x, y, z$ are Pauli matrices acting on the h-th qubit, $\alpha_a \geq 0$ and $\sum_a \alpha_a = 1$. Here the goal is to estimate the frequency ω with the best possible precision. The latter satisfies the Quantum Cramer-Rao bound

$$\delta^2\omega\ T \geq (QFI/t)^{-1}$$

where the limits to the precision depend on the instant of time t in which the estimation takes place. There are two extremal cases of noise relevant for the discussion. In the "parallel" case ($\alpha_z = 1$) the noise acts locally on each qubit along the z direction. In the "transverse" case ($\alpha_x = 1$) the noise acts locally on each qubit along the x direction.

In [66] the Authors showed, in agreement with [75], that in the parallel case one can only achieve a precision that scales in terms of the number of qubits at most as M^{-1}, the SQL. On the other hand, the Authors were able to show that, in the purely transverse case, by optimizing the time at which the estimation takes place such that $t_{opt} = (3/\gamma\omega^2 M)^{1/3}$, then one can overcome the SQL and obtain $\delta^2\omega\ T \geq M^{-5/3}$. In the following we show how these two cases can be cast in our framework.

In order to analyze the dynamics we use the same TPS^R Eq. (31) defined for the noise-free GHZ case in Section (III E). Indeed, as we show in Appendix VI, for any direction of the noise (i.e., for any value of $\alpha_x, \alpha_y, \alpha_z$) the state of the M-qubit system $\rho_{\omega,\gamma}(t)$, solution of the master equation, can be written in TPS^R representation as

$$\rho_{\omega,\gamma}(t) = \sum_k p_k(t)\tau_k(\omega,t)\tilde{\otimes}|k\rangle\langle k| \tag{48}$$

where: $\tau_k(\omega,t)$ are single-qubit density matrices ; $p_k(t)$ is the probability of finding the system in the k-sector and it does not depend on ω (for a detailed expression of $\rho_{\omega,\gamma}(t)$ see Appendix VI or Ref. [66]). Before continuing our analysis we notice that, in principle the SLD $L_\omega(t)$ is a function of time and so are its eigenvectors. The latter can be numerically evaluated and the relative $TPS^R(t)$ determined. The procedure is in general involved. However, having realized that in $TPS^R = TPS^R(0)$ (31) the state (48) has a very simple structure, we proceed within this picture. Its usefulness can be further justified by noticing that, since in general $t_{opt} \ll 1$, one may argue that the dynamics taking place in the single qubit factor identified by $TPS^R(0)$ well represents the one taking place in the actual single qubit factor defined by $TPS^R(t_{opt})$.

Thanks to the introduced representation, we can first notice that the coherent dynamics only affects the single qubit subsystem $\mathcal{H}_2$. The state of the system $\rho_{\omega,\gamma}(t)$ has, for any (t,ω), the generic form introduced in Section III D. Therefore

$$QFI(t,\omega) = \sum_k p_k(t)QFI_k(t,\omega) = \langle QFI_k\rangle(t,\omega) \tag{49}$$

i.e., the QFI is the average of the single qubit QFI_k (in the following we adopt the simplified notation $\sum_k p_k f_k \to \langle f_k\rangle$). Equation (49) is valid for all kind of noise directions (i.e., all allowed values of $\alpha_x, \alpha_y, \alpha_z$) and it neatly shows the qubit-like nature of the estimation process, allowing us to investigate the connection between the QFI and single qubit coherences.

We start by focusing on the case of "parallel" noise. When $\alpha_z = 1, \alpha_y = \alpha_x = 0$, the coherent and decoherent part of the evolution commute and furthermore, since $|\psi_0\rangle = |GHZ_0^+\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)\tilde{\otimes}|0\rangle$, the evolution takes place entirely in the $k=0$ sector. The master equation can then be reduced to a master equation for τ_0 as (see Appendix VI)

$$\partial_t\tau_0 = -\frac{iM\omega}{2}[S_x,\tau_0] - \frac{M\gamma}{2}[\tau_0 - S_x\tau_0 S_x] \tag{50}$$

where S_x is the Pauli operators acting on the single qubit factor $\mathcal{H}_2$ as $S_x|\pm\rangle = |\mp\rangle$. Just as in the noiseless GHZ case, the estimation process is a single qubit one and our approach applies. In particular, the eigenvalues of τ_0 depend on (t,γ) but not on ω, and $f(\tau_0(\omega)) = 0$; therefore equation (44) reduces to

$$-\left(\partial^2_{\omega_1}Coh\right)_{\omega_1=\omega} = QFI^Q. \tag{51}$$

The Quantum Fisher Information $QFI(t,\omega) = QFI_{k=0}(t,\omega) = -\left(\partial^2_{\omega_1}Coh\left[\tau_0(t,\omega_1)\right]\right)_{\omega_1=\omega}$ can be written for all (t,ω) as a second order variation of the coherence of the measurement basis. The TPS^R picture allows to understand why in the "parallel" case the estimation precision cannot beat the SQL. While the phase imprinted, $M\omega$, is proportional to the number of qubits, the whole decoherence acts on the single qubit with a strength $M\gamma$ also proportional to the number of qubits, thus neutralizing the enhancement in the precision provided by $M\omega$. The same result can be explained by noticing that in the parallel case the coherent (46) and the decoherent (47) map commute, and therefore the estimation process is therefore fully equivalent to a single qubit multiround one enacted on a highly decohered state.

We now pass to analyze the "transverse" noise case. Its relevance stems from the fact that in [66] it was shown that it is possible to attain a precision in the estimation of ω that scales with the number of qubits as $M^{-5/3}$, i.e. to attain a quasi-Heisenberg scaling. When $\alpha_x = 1, \alpha_y = \alpha_z = 0$ the coherent and decoherent part of the state evolution no longer commute. In order to discuss the role of coherence in the estimation process and to investigate the connection with single qubit coherences we then have to use formula (44). Therefore in general one expects that $QFI_k \neq -\partial^2_\omega Coh\left[\tau_k(t=t_{opt},\omega)\right]$. However, our numerical results show that the behavior of the terms in (44) is the following. For fixed $(t=t_{opt})$ the term $\left\langle QFI_k^Q\right\rangle$ grows as $M^{5/3}$ and it has the same scaling as $\langle QFI_k\rangle$. On the other hand , the remaining contributions in Eq. (43) do not scale with M and remain small:

$$\left|\left\langle f_k^\epsilon(\epsilon_i^\lambda)\right\rangle\right|, \left|\langle QFI_k^c\rangle\right|, \left|\langle f^\chi(\rho_k)\rangle\right| < 1 \ll \left\langle QFI_k^Q\right\rangle \tag{52}$$

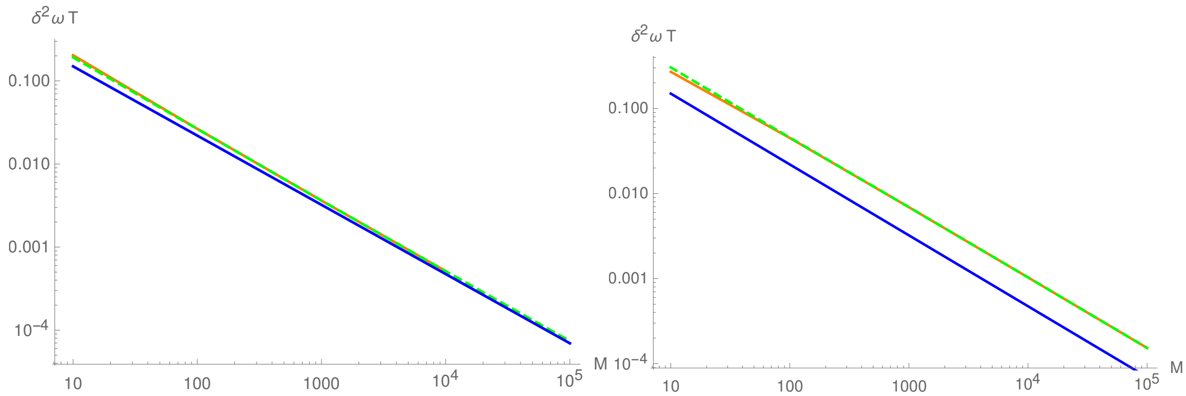

Figure 1: **(a)** $\left(\langle QFI_k \rangle / t_{opt}\right)^{-1}$ (orange) and $-\left(\left\langle \partial_\omega^2 Coh\left[\tilde{\tau}_k\right]\right\rangle / t_{opt}\right)^{-1}$(green) as a function of the number of qubits M, for $\omega = \gamma = 1$. The blue curve is the predicted bound $\left(QFI/t_{opt}\right)^{-1} = (9/8\ \gamma\omega^2)^{1/3} N^{-5/3}$**(b)** $\left(QFI_\xi/t_{opt}\right)^{-1}$ (orange) and $\left(-\partial_\omega^2 Coh\left[\xi_\lambda\right]/t_{opt}\right)^{-1}$(green) of the reduced qubit state ξ_λ as a function of the number of qubits M, for $\omega = \gamma = 1$. The blue curve is the precision bound $\left(QFI/t_{opt}\right)^{-1} = (9/8\ \gamma\omega^2)^{1/3} N^{-5/3}$.

Therefore, for large M, we have:

$$QFI \approx \left\langle -\partial_\omega^2 Coh\left[\tau_k(\omega, t = t_{opt})\right]\right\rangle \approx \left\langle QFI_k^Q \right\rangle . \tag{53}$$

In Fig. 1(a) we present, for $\gamma = \omega = 1$, the scaling with M of $\left(\left\langle QFI_k(\omega, t = t_{opt})\right\rangle / t_{opt}\right)^{-1}$ and $\left(\left\langle -\partial_\omega^2 Coh\left[\tau_k(\omega, t = t_{opt})\right]\right\rangle / t_{opt}\right)^{-1}$, together with the asymptotic scaling $\left(QFI/t_{opt}\right)^{-1} = (9/8\ \gamma\omega^2)^{1/3} N^{-5/3}$ that can be predicted with the channel extension method [66]. As shown in the plots, the two quantities have the same asymptotic behavior leading to the expected super-SQL scaling on the precision $\delta^2\omega\ T$. This result neatly shows that, in the transverse case, what matters for the achievement of a super-SQL scaling of the estimation precision is the second order variation of single qubit coherences. If we now look at the state (48) we see that the single qubit subsystem $\mathcal{H}_2$, the only one relevant for the estimation process, is never entangled with the subsystem $\mathcal{H}_{2^{M-1}}$ during the whole evolution.

In order to grasp how the estimation process physically works we give the following explanation. By using the description of the solution given by Eq. (48) and by projecting the master equation onto the k subspaces (see the Appendix VI) one obtains a system of coupled differential equations

$$\partial_t \tilde{\tau}_k = \mathcal{H}_k + \mathcal{L}_k \tag{54}$$

that is equivalent to the master equation. For each of the (un-normalized) single qubit density matrices $\tilde{\tau}_k(\omega, t) = p_k(t)\tau_k(\omega, t)$ one has that the coherent part of the evolution is dictated by

$$\mathcal{H}_k \ = \ -\frac{i\omega}{2}\left(M - 2|k|\right)\left[S_x, \tilde{\tau}_k\right] \tag{55}$$

where $|k|$ is the number of ones in the binary representation of k, and the decoherent part by

$$\mathcal{L}_k \ = - \ \frac{\gamma}{2}\left[M\tilde{\tau}_k - S_z\tilde{\tau}_{k'(M)}S_z - \sum_{h=1}^{M-1}\tilde{\tau}_{k'(h)}\right] \tag{56}$$

where $k'(h)$ is an h-dependent permutation of the k sectors (see Appendix VI for details). The role of the decoherent part (56) is to couple different sectors: each k is coupled to the sectors $k'\ (h)$ whose binary representation is obtained by flipping just one of the M bits in k.
In order to study how the process evolves we focus on the case $\omega = \gamma = 1$. Numerical results show that the system of equations (54) is approximately solved by the following ansatz. Writing the unnormalized single qubit states as

$$\tilde{\tau}_k = p_k(t)\left(\frac{\mathbb{I}_2 + h_k\hat{n}_k \cdot \vec{S}}{2}\right)$$

where $h_k \hat{n}_k$ are the corresponding Bloch vectors (lying on the yz plane in TPS^R representation), an approximate solution is obtained by replacing each $\hat{n}_k$ with the ansatz

$$\hat{a}_k = (0, \sin[(m_k - 1)\omega t], \cos[(m_k - 1)\omega t]) \quad (57)$$

where $m_k = max(|k|, M - |k|)$. Thus, the overall process can be pictured as follows. The evolution starts in the $k = 0$ sector with the initial state $|\psi_0\rangle = |GHZ_0^+\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)\tilde{\otimes}|0\rangle$. The result of the evolution is given by the interplay of the coherent $\mathcal{H}_k$ (55) and decoherent $\mathcal{L}_k$ (56) part. The action of the "transverse" decoherence is twofold. On one hand it progressively populates the sectors $k \in K(t) = \{k : m_k \lesssim M)\}$. Once these sectors are populated, the corresponding states undergo the action of the coherent evolution $\mathcal{H}_k$. As it can be inferred from the approximate Bloch vector (57), the latter is similar to the multi-round single qubit evolution described in Section III E. The actual phase (frequency ω) imprinted onto the state is $(m_k - 1)\omega \lesssim M\omega$. The process continues up to t_{opt} where one can check that only the sectors $k \in K(t)$ are substantially populated. During the evolution each $k \in K(t)$ state undergoes a decoherence process. However the choice of $t_{opt} \ll 1$ guarantees that: $i)$ the effect of decoherence is not too relevant; $ii)$ the multi round single qubit processes has sufficient time to impress the phase $(m_k - 1)\omega \lesssim M\omega$ onto each single qubit state. Should the measurement time be $\ll t_{opt}$ the latter effect would not be seen.

While this qualitative picture is based on an approximate solution devised for $\omega = \gamma = 1$, one can find similar solutions for other values of ω, γ. The relevant point is that the above discussion shows how the overall estimation process can be seen as a *parallel multi round estimation one,* enacted onto (a fraction) of the single qubits $\tilde{\tau}_k$ such that $k \in K(t)$. In terms of the overall QFI, each relevant sector contributes with a QFI_k that is fairly well approximated by the second order derivative of the relative coherence function:

$$QFI_k \approx QFI_k^Q \approx -\partial_\omega^2 Coh\left[\tau_k(\omega, t = t_{opt})\right]$$

Since in each relevant sector the process is a multi-round single qubit one, the super-SQL scaling is the result of a change of the coherence of the eigenbasis of the SLD (L_k) pertaining to each sector.

We conclude this section by analyzing how possible bounds on QFI that can be obtained within the TPS^R used. In Section III B we have seen that given a TPS^R one has

$$FI_2 \leq \; QFI(Tr_{\mathcal{H}_{N/2}}\left[\rho_{(t,\omega)}\right]) \; \leq QFI(\rho_{(t,\omega)}).$$

Now if one aims at computing FI_2 in principle one needs to determine the $TPS^R(t)$ induced by the time dependent SLD $L_\omega(t)$. However, as noticed above the $TPS^R(0)$ we have used so far not only is easier to use but it allows to analytically compute $\xi(t, \omega) = Tr_{\mathcal{H}_{N/2}}[\rho_\lambda]$ and $QFI_\xi(t, \omega)$. The latter turns out to be a meaningful lower bound to $QFI(\rho_{(t,\omega)})$. Indeed, as shown in Fig. 1(b) one can verify that also in this case

$$\left(QFI_\xi(t = t_{opt}, \omega)/t_{opt}\right)^{-1} \approx \left(-\partial_\omega^2 Coh\left[\xi(t = t_{opt}, \omega_1)\right]/t_{opt}\right)^{-1}_{\omega_1 = \omega}$$

and the precision that can be obtained by using $\xi(\omega, t)$ as probe state allows to attain the usual super-SQL scaling with M. This result can be neatly interpreted as follows. If one traces over $\mathcal{H}_{2^{M-1}}$ the general master equation, which amounts to sum over k the equations (54) , one obtains the following equation for $\xi(t, \omega)$

$$\partial_t \xi(t, \omega) = \frac{-i\omega}{2}\sum_k (M - 2|k|)\left[S_x, \tilde{\tau}_k\right] + \frac{\gamma}{2}\left(\xi(t, \omega) - S_z \xi(t, \omega) S_z\right).$$

If now supposes that M is large enough one can numerically check that the sectors involved by the evolution on the time scale given by t_{opt} are only those such that $|k| \ll M$ then the previous equation can be approximated by

$$\partial_t \xi(t, \omega) \approx \frac{-i\omega}{2} M\left[S_x, \xi(t, \omega)\right] + \frac{\gamma}{2}\left(\xi(t, \omega) - S_z \xi(t, \omega) S_z\right)$$

and one directly sees that, while the coherent part acts with MS_x the decoherence part is only proportional to $\gamma/2$. While the latter is just a rough approximation, it gives an intuitive way to understand why the estimation shows a super-SQL scaling: compared to the parallel case, on the time scale given by t_{opt} the single qubit is only marginally touched by decoherence.

V. COHERENCE, QUANTUM PHASE TRANSITIONS AND ESTIMATION

We finally analyze the case of criticality-enhanced quantum estimation processes. The theory is based on the fidelity approach to QPTs [38]. The scenario is the following: suppose $H_\lambda = H_0 + \lambda V,\ \lambda \in \mathbb{R}$ is a family of Hamiltonians.

The corresponding manifold of ground states $\left\{|0^\lambda\rangle\right\}_\lambda$ of H_λ can be adiabatically generated by means of the unitary operator [37]

$$O_\lambda = \sum_k |n^{\lambda+\delta\lambda}\rangle\langle n^\lambda|$$

where $\left\{|n^\lambda\rangle, E_n^\lambda\right\}$ are the eigenstates and the corresponding eigenvalues of H_λ. The QFI for a given λ reads[19]:

$$QFI(\lambda) = 4\sum_{n>0}\frac{\left|\langle n^\lambda|V|0^\lambda\rangle\right|^2}{\left(E_0^\lambda - E_n^\lambda\right)^2} = 4g_\lambda^{FS} \tag{58}$$

and the estimation precision is proportional to the Fubini-Study metric. Suppose the system described by H_λ undergoes a QPT when $\lambda = \lambda_c$. If one aims at estimating the parameter λ with the highest possible precision, in proximity of the critical point λ_c one can exploit the scaling behavior of the QFI with respect the size of the system L. The scaling is determined by the critical exponents that define the given QPT [19, 38]. Indeed

$$g_\lambda \sim L^{-\nu\Delta_Q+d}$$

where $\Delta_Q = 2\Delta_V - 2\zeta - d$ is a function of the scaling exponent Δ_V of the operator V that drives the QPT, the dynamical exponent ζ and the scaling exponent of the correlation length ν. If Δ_V is such that $\Delta_Q < 0$ i.e., if V is "sufficiently" relevant, g_λ scales in a super-extensive way and so does the QFI, thus allowing for an enhancement of the estimation precision. If V is "insufficiently" relevant i.e., Δ_V is such that $\Delta_Q > 0$, as for example in Berezinskii-Kosterlitz-Thouless type of QPTs, one cannot take advantage of the super-extensive behaviour to estimate λ.
It is easy to show that Equation (58) is a particular case of (11). Indeed, given $|0^{\lambda+\delta\lambda}\rangle = O_\lambda|0^\lambda\rangle$ i.e., the ground state for $\lambda+\delta\lambda$, using perturbative expansion $|0^{\lambda+\delta\lambda}\rangle \approx |0^\lambda\rangle + |\vec{v}^\lambda\rangle$, one writes the the first order correction in $\delta\lambda$ as

$$|\vec{v}^\lambda\rangle = \sum_{n>0}\frac{\langle n^\lambda|V|0^\lambda\rangle}{\left(E_0^\lambda - E_n^\lambda\right)}|n^\lambda\rangle$$

with $|\hat{v}^\lambda\rangle = |\vec{v}^\lambda\rangle/|\vec{v}|$. Consider now the following orthonormal basis

$$\mathcal{B}_{0,v} = \left\{\left(|0^\lambda\rangle \pm |\hat{v}^\lambda\rangle\right)/\sqrt{2}\right\}\bigcup\{|\alpha_n\rangle\}_{n=2}^{L^d}$$

where $\{|\alpha_n\rangle\}_{n=2}^{L^d}$ is a generic set of orthonormal vectors. Then, as proven in Appendix VII,

$$4g_\lambda^{FS} = \left[-\partial_{\delta\lambda}^2 Coh_{\mathcal{B}_{0,v}}\left(|0^\lambda\rangle\right)\right]_{\delta\lambda=0} \tag{59}$$

i.e., the metric coincides with the second order variation of the coherence of $\mathcal{B}_{0,v}$ with respect to the ground state $|0^\lambda\rangle$. In accordance with (11) the first consequence of the previous result is that the geometry of the manifold of ground states is determined by the coherence properties of $\mathcal{B}_{0,v}$. Secondly, the non-analiticities and scaling properties of g_λ^{FS}, that on one hand signal the presence of a QPT and on the other hand are at the basis of the CEQE, are those pertaining to the physical quantity $Coh_{\mathcal{B}_{0,v}}\left(|0^\lambda\rangle\right)$, and in particular its second order variation. The latter is single-qubit in nature since it pertains the subspace $span\left\{|0^\lambda\rangle, |\hat{v}^\lambda\rangle\right\}$.

Quantum Phase transitions can in general be signaled by several different properties of the underlying system. For example, by focusing on subsystems such as one- or two-site density matrices for spin chains, one can find several QPTs signature by analyzing the non-analiticities of correlations and coherence measures [36, 59–62]. Our approach instead focuses on the single qubit subspace $span\left\{|0^\lambda\rangle, |\hat{v}^\lambda\rangle\right\}$ and on the variation of the relevant coherence impressed by $O_\lambda(V)$. The fidelity approach fails to signal the QPTs and. correspondingly the criticality does not allow to enhance the estimation precision in CEQE, whenever the operator V is "insufficiently" relevant i.e., Δ_V is such that $\Delta_Q > 0$. In our picture this can be interpreted as the consequence of the fact that the variation of the relevant coherence impressed by $O_\lambda(V)$ is in these cases too weak and one cannot take advantage of the super-extensive behavior to estimate λ [19].

VI. CONCLUSIONS

Coherence is one of the fundamental features that distinguish the quantum from the classical realm. The perspective adopted in this work allows to link coherence (its second order variation in a specific basis) to the geometry of quantum states and their statistical distinguishability, and thus to the Quantum Fisher Information. The connection allows to establish a framework that encompasses a wide variety of single parameter estimation processes: noiseless and noisy quantum phase estimation based on single/multi-qubit probes, and criticality enhanced quantum estimation. Overall our findings show how to quantify the notion that coherence is the resource that must be engineered, controlled and preserved in these quantum estimation processes.

As for quantum phase estimation, the use of specific factorizations of the underlying quantum system, i.e., specific tensor product structures, allows to express the Quantum Cramer-Rao bound to the estimation precision in terms of two contributions: the Fisher Information of a single qubit; and the second order variation of the classical correlations between the observables defined by the main object of the theory i.e., the Symmetric Logarithmic Derivative. The adopted perspective thus allows to discuss the role of (quantum) correlations in estimation processes. In several relevant cases (quantum) correlations in the state probe are not intrinsically required and the estimation is effectively equivalent to a process based on a single qubit interacting with a possibly much larger system. In particular we show how various relevant protocols based on different strategies, such as multi-round application of phase shifts to a single qubit or protocols based on pure and mixed GHZ and NOON states, are formally equivalent and are based on the exploitation of the very same resource: the variation of a single qubit coherence. In doing so we provide an example of M-qubit based evolution in which the Heisenberg limit in the estimation precision can be attained with the use of an uncorrelated M-qubit probe state.

As for noisy estimation processes, we have focused on a prototypical example based on GHZ states that achieves a quasi-Heisenberg scaling of the precision. We have discussed the protocol and we have shown how within our perspective: *i*) the estimation procedure can be described as a parallelization of the single qubit multi-round strategy, where the quasi-Heisenberg scaling is rooted in single-qubit coherences variations; *ii*) one can analytically derive, even for complex multi-qubit noisy evolutions, meaningful lower bounds to the Quantum Fisher Information that allow to infer its scaling behaviour. The approach is suitable to be extended and applied to other relevant noisy estimation processes.

We have finally discussed criticality-enhanced quantum estimation processes. In doing so we have recognized the role of a specific kind of (global) coherence in quantum phase transitions. The non-analiticities of such coherence are at the basis of the sensitivity scaling of criticality-enhanced estimation protocols and they correspond to the global signatures of zero-temperature phase transitions found within the fidelity approach.

While in laying down our framework we have focused on particular kinds of single-parameter estimation protocols, our approach is suitable to be extended to more general estimation processes.

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I. FUBINI-STUDY METRIC AND COHERENCE

We show below how the Fubini-Study (FS) metric can be related to variation of the coherence of a generic eigenbasis $\mathcal{B}^{\lambda}_{\boldsymbol{\alpha}}$ of L_λ. Suppose one has a one-parameter family of states $\{|\psi_\lambda\rangle \in \mathbb{C}^N,\ \lambda \in \mathcal{I} \subset \mathbb{R}\}$ and one wants to estimate λ. The estimation problem was solved in [6] as follows. Assume that the states $|\psi_\lambda\rangle$ are normalized, and the curve $|\psi_\lambda\rangle$ in $\mathcal{H}_N$ is of class C_2. Then, in a infinitesimal neighborhood $\lambda + d\lambda$ around to the generic λ, one can expand

$$|\psi_{\lambda+d\lambda}\rangle = |0\rangle + |v\rangle d\lambda + |w\rangle d\lambda^2 + \mathcal{O}(d\lambda^3) \tag{I.1}$$

with $|0\rangle \equiv |\psi_\lambda\rangle$, $|v\rangle \equiv \left(\frac{d}{d\lambda}|\psi_{\lambda+d\lambda}\rangle\right)_{d\lambda=0}$ and $|w\rangle \equiv \left(\frac{d^2}{d\lambda^2}|\psi_{\lambda+d\lambda}\rangle\right)_{d\lambda=0}$. In Ref. [6], it was shown that the SLD in $d\lambda = 0$ can be written as

$$L_\lambda = |0\rangle\langle v^\perp| + |v^\perp\rangle\langle 0| \tag{I.2}$$

where $|v^\perp\rangle = |v\rangle - \langle 0|v\rangle|0\rangle$, and one gets the quantum Fisher information (QFI)

$$QFI = 4\langle v^\perp|v^\perp\rangle = 4\big(\langle v|v\rangle - |\langle v|0\rangle|^2\big) \tag{I.3}$$

which is seen to coincide with the Fubini-Study metric[40]. The latter provides both the geometric distance and a measure of the statistical distinguishability of two infinitesimally closed pure states. Due to the form of L_λ the estimation problem pertains a single qubit subspace $\mathcal{H}_2 = span\left\{|0\rangle, |v^\perp\rangle\right\}$. The optimal measurement basis that allows to attain QFI is uniquely defined only in $\mathcal{H}_2$ where

$$|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm \frac{1}{\langle v^\perp|v^\perp\rangle^{1/2}}|v^\perp\rangle) \tag{I.4}$$

give the eigenbasis $\mathcal{B}_\pm = \{|\pm\rangle\}$ of L_λ pertaining to the only non zero eigevalues $\pm\left|\langle v^\perp|v^\perp\rangle\right|$ (in the following we drop for simplicity the eigenvectors' dependence on λ). Suppose now we choose a generic basis for the kernel of L_λ $\mathcal{B}_{Ker} = \{|n\rangle\}_{n=3}^N$ such that $\langle n|\pm\rangle = 0,\ \forall n$. Then for all bases $\mathcal{B}^\lambda_{\boldsymbol{\alpha}} = \mathcal{B}_\pm \bigcup \mathcal{B}_{Ker}$ of the whole of $\mathcal{H}_N$ we have that the probabilities $p_\pm^{\lambda+d\lambda} = \left|\langle\pm|\psi^{\lambda+d\lambda}\rangle\right|^2$ evaluated up to order $\mathcal{O}(d\lambda^2)$ read

$$p_\pm^{\lambda+d\lambda} = \frac{1}{2}(1 \pm 2\langle v^\perp|v^\perp\rangle^{1/2}d\lambda \pm 2Re\frac{\langle w|v^\perp\rangle}{\langle v^\perp|v^\perp\rangle^{1/2}}d\lambda^2 + \mathcal{O}(d\lambda^3)) \tag{I.5}$$

where we have used $\langle v|v^\perp\rangle = \langle v^\perp|v^\perp\rangle$ and the conditions $Re[\langle v|0\rangle] = 0$ and $2Re\langle w|0\rangle = -\langle v|v\rangle$ (implied by the normalization condition $\langle\psi_\lambda|\psi_\lambda\rangle = 1$ at first and second order in $d\lambda$). Thus, we obtain

$$\left(p_\pm^{\lambda+d\lambda}\right)_{d\lambda=0} = p_\pm^\lambda = 1/2, \quad \left(\partial_{d\lambda}p_\pm^{\lambda+d\lambda}\right)_{d\lambda=0} = \pm\langle v^\perp|v^\perp\rangle^{1/2}, \quad \left(\partial^2 p_\pm^{\lambda+d\lambda}\right)_{d\lambda=0} = \pm Re\frac{\langle w|v^\perp\rangle}{\langle v^\perp|v^\perp\rangle^{1/2}} \tag{I.6}$$

As for $\mathcal{B}_{Ker}$ one has $p_n^{\lambda+d\lambda} = |\langle\psi^\lambda|n\rangle|^2 = \mathcal{O}(d\lambda^4)$ and what matters is that they are $o(d\lambda^2)$. Consequently, if one considers the coherence function $Coh_{\mathcal{B}_{\boldsymbol{\alpha}}}\left(|\psi_\lambda\rangle\right)$ for a generic $\mathcal{B}^\lambda_{\boldsymbol{\alpha}}$ one has the two relations:

$$\left[\partial_{d\lambda}Coh_{\mathcal{B}^\lambda_{\boldsymbol{\alpha}}}\left(|\psi_{\lambda+d\lambda}\rangle\right)\right]_{d\lambda=0} = 0 \tag{I.7}$$

$$-\left[\partial^2_{d\lambda}Coh_{\mathcal{B}^\lambda_{\boldsymbol{\alpha}}}\left(|\psi_{\lambda+\lambda}\rangle\right)\right]_{d\lambda=0} = \sum_{i=\pm}\frac{\left(\partial_{d\lambda}p_\pm^{\lambda+d\lambda}\right)_{d\lambda=0}}{p_i^\lambda} = QFI \tag{I.8}$$

while $f\left(|\psi_\lambda\rangle\langle\psi_\lambda|\right) = \sum_{i=\pm}\left(\partial^2 p_i^{\lambda+d\lambda}\right)_{d\lambda=0}\log_2 p_i^\lambda = 0$. Therefore the FS metric can in general be expressed as a curvature of the coherence $Coh_{\mathcal{B}^\lambda_{\boldsymbol{\alpha}}}$ of a generic eigenbasis $\mathcal{B}^\lambda_{\boldsymbol{\alpha}}$ of L_λ with respect to $|\psi_\lambda\rangle$ around a maximum. Thus, in terms of coherence, what matters for the estimation process and for the statistical distinguishability between two neighboring pure states is indeed the variation of the coherence within the single-qubit subspace $\mathcal{H}_2$ spanned by $\mathcal{B}_\pm = \{|\pm\rangle\}$.

II. RESULTS FOR $N = 2$

In this section, we derive the results presented in the main text for the qubit case $N = 2$. Without loss of generality we choose the single qubit state

$$\rho_0 = (1 + \vec{z}\cdot\boldsymbol{\sigma})/2 \tag{II.1}$$

where $\vec{z} = z\hat{z} = z(0,0,1)$, $0 \le z \le 1$ and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli matrices. The phase generator is

$$G = \gamma\left(\hat{\gamma}\cdot\boldsymbol{\sigma}\right) \tag{II.2}$$

with $\hat{\gamma} = (\sin\delta, 0, \cos\delta)$, such that its eigenbasis lies in the $\hat{x}\hat{z}$ plane, forming an angle $0 \le \delta \le \frac{\pi}{2}$ with $\hat{z}$. The strength of G is measured by its norm $Tr[G^2] = 2\gamma^2$, where $\gamma > 0$. A generic measurement basis $\mathcal{B}_{\hat{b}}$ is defined by the projectors

$$\Pi_\pm^{\hat{b}} = (1 \pm \hat{b}\cdot\boldsymbol{\sigma})/2 \tag{II.3}$$

with $\hat{b} = \{\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta\}$ a generic Bloch vector. The state

$$\rho_\lambda = e^{-i\lambda\vec{\gamma}\cdot\boldsymbol{\sigma}}\rho_0 e^{i\lambda\vec{\gamma}\cdot\boldsymbol{\sigma}} \tag{II.4}$$

is given as $\rho_\lambda = (1 + \vec{z}_\lambda\cdot\boldsymbol{\sigma})/2$ with

$$\vec{z}_\lambda = \cos 2\gamma\lambda\ \vec{z} + \sin 2\gamma\lambda\ (\vec{z}\times\vec{\gamma}) + (1 - \cos 2\gamma\lambda)\vec{\gamma}(\hat{\gamma}\cdot\vec{z}) \tag{II.5}$$

Thus, the probabilities $p^{\hat{b}}_{\pm}(\lambda) = Tr[\rho_\lambda \Pi^{\hat{b}}_{\pm}]$ are obtained as

$$p^{\hat{b}}_{\pm}(\lambda) = 1/2Tr[(1 \pm \hat{b}\cdot\boldsymbol{\sigma})\rho_\lambda] = 1/2\Big(1 \pm \hat{b}\cdot\vec{z}_\lambda\Big) \tag{II.6}$$

Their derivatives are computed as

$$\partial_\lambda p^{\hat{b}}_{\pm}(\lambda) = \pm 1/2\hat{b}\cdot\partial_\lambda\vec{z}_\lambda, \qquad \partial^2_\lambda p^{\hat{b}}_{\pm}(\lambda) = \pm 1/2\hat{b}\cdot\partial^2_\lambda\vec{z}_\lambda \tag{II.7}$$

with $\partial_\lambda\vec{z}_\lambda = 2\gamma(-\sin 2\gamma\lambda\ \vec{z} + \cos 2\gamma\lambda\ (\vec{z}\times\vec{\gamma}))$ and $\partial^2_\lambda\vec{z}_\lambda = 4\gamma^2(-\cos 2\gamma\lambda\ \vec{z} - \sin 2\gamma\lambda\ (\vec{z}\times\hat{\gamma}))$. In $\lambda = 0$, one gets

$$p^{\hat{b}}_{\pm}(\lambda = 0) = \ 1/2\Big(1 \pm \hat{b}\cdot\vec{z}\Big) \tag{II.8}$$

$$(\partial_\lambda p^{\hat{b}}_{\pm})_{\lambda=0} = \ \pm\gamma(\vec{z}\times\hat{\gamma})\cdot\hat{b} \ = \pm(\vec{z}\times\vec{\gamma})\cdot\hat{b} \tag{II.9}$$

$$(\partial^2_\lambda p^{\hat{b}}_{\pm})_{\lambda=0} = \ \pm 2\gamma^2\ \hat{b}\cdot\vec{z} \tag{II.10}$$

The Fisher Information for $\mathcal{B}_{\theta,\phi}$ in $\lambda = 0$ is computed as

$$\begin{aligned} FI(\mathcal{B}_{\hat{b}}, \rho_0, G) = &\ \frac{((\partial_\lambda p^{\hat{b}}_{+})_{\lambda=0})^2}{p^{\hat{b}}_{+}(\lambda=0)} + \frac{((\partial_\lambda p^{\hat{b}}_{-})_{\lambda=0})^2}{p^{\hat{b}}_{-}(\lambda=0)} = \\ = &\ 2((\vec{z}\times\vec{\gamma})\cdot\hat{b})^2\Bigg(\frac{1}{\Big(1+\hat{b}\cdot\vec{z}\Big)} + \frac{1}{\Big(1-\hat{b}\cdot\vec{z}\Big)}\Bigg) = 4\frac{(\vec{\gamma}\times\vec{z}\cdot\hat{b})^2}{1-(\vec{z}\cdot\hat{b})^2} \end{aligned}$$

In terms of θ, ϕ and δ the latter can be written as

$$FI(\mathcal{B}_{\theta,\phi}, \rho_0, G) = 4\gamma^2 z^2 \sin^2\delta\frac{\sin^2\theta\sin^2\phi}{1 - z^2\cos^2\theta}$$

For mixes states $z < 1$, the latter is maximized when $\hat{z}, \hat{\gamma}, \hat{b}$ form an orthogonal triple. The maximization over $\hat{b}$ can be performed by finding the maximum with respect to θ, ϕ . The maximum over ϕ is obviously $\phi = \pi/2$, while the maximum over θ can be easily found by computing the critical points of $\frac{\sin^2\theta}{1-z^2\cos^2\theta}$, which gives $\theta = \pi/2$. Therefore, the maximization over $\hat{b}$ results in the choice $\hat{b} = \{0, 1, 0\} \propto \hat{\gamma}\times\hat{z}$. In turn, the symmetric logarithmic derivative in $\lambda = 0$ can be shown to be

$$L_0 = -2(\hat{\gamma}\times\hat{z})\cdot\boldsymbol{\sigma} \tag{II.11}$$

Indeed, one has

$$\partial_\lambda\rho_\lambda|_{\lambda=0} = -i[G, \rho_0] = \vec{\gamma}\times\vec{z}\cdot\boldsymbol{\sigma} \tag{II.12}$$

and one can immediately verify that $\frac{1}{2}(L_0\varrho_o + \varrho_o L_0) = \vec{\gamma}\times\vec{z}\cdot\boldsymbol{\sigma}$. Thus, the eigenbasis of L_0 corresponds to $\hat{\alpha} = \vec{\gamma}\times\vec{z}/|\vec{\gamma}\times\vec{z}| = \{0, 1, 0\}$, which coincides with the optimal measurement.

Furthermore, from the above formulas (II.8,II.9,II.10), when considering the coherence function $Coh_{\mathcal{B}_\alpha}(\rho_\lambda) = -\mathcal{V}(\rho_\lambda) + \sum_{i=\pm} p^{\hat{\alpha}}_i \log p^{\hat{\alpha}}_i$, one obtains with some simple algebra

$$[\partial_\lambda Coh_{\mathcal{B}_\alpha}(\rho_\lambda)]_{\lambda=0} = 0, \qquad QFI = -\left(\partial^2_\lambda Coh_{\mathcal{B}_\alpha}(\rho_\lambda)\right)_{\lambda=0} \tag{II.13}$$

The above results hold in particular for the limiting case of pure states ($z = 1$), if one measures on the eigenbasis $\mathcal{B}_\alpha$ of the SLD. The latter is not the only basis that allows to attain the QFI. Indeed, for pure states the Fisher information for a generic measurement basis is independent of the angle θ and reads

$$FI(\mathcal{B}_{\theta,\phi}, \rho_0, G) = 4\gamma^2 z^2 \sin^2\delta\sin^2\phi$$

such that the bound QFI can in principle be achieved by any basis such that $\phi = \pi/2$. However, such bases in fact not all equivalent. Indeed the logic of the estimation process as described within the Crámer-Rao formalism is the following. In general one needs to know in advance with some precision the value of λ. This can be achieved with a (non-optimal) pre-estimation process on a subset of the probes, which leads to a value λ_{est}. Then one applies to the initial state the shift $U_{\delta\lambda} = \exp -i\delta\lambda G$ with $\delta\lambda = \lambda - \lambda_{est} \ll 1$. Only then the choice of the optimal measurement basis becomes meaningful; in particular, if one supposes that $\delta\lambda = 0$ the actual precision for a generic basis $\mathcal{B}_{\hat{b}}$ is given by

$$F(\mathcal{B}_{\hat{b}}, |\psi_{\delta\lambda}\rangle\langle\psi_{\delta\lambda}|) \approx 4\gamma^2 \sin^2\phi - \gamma^3 \left(16\cos^2\phi \sin\phi \cot\theta\right) \delta\lambda. \tag{II.14}$$

The latter now *does depend on* θ, and if ϕ is only approximately equal to $\pi/2$, for example due to imprecision in the measurement apparatus, the bases corresponding to different values of θ are no longer equivalent. For example, if $\theta \approx 0, \pi$ it can happen that $F(\mathcal{B}_{\theta,\phi}, |\psi_{\delta\lambda}\rangle\langle\psi_{\delta\lambda}|) \ll QFI$. Instead, for $\theta \approx \pi/2$ i.e., $\mathcal{B}_{\hat{b}} \approx \mathcal{B}_{\hat{\alpha}}$, this problem can be avoided. The choice of $\hat{b} = \hat{\alpha}$ becomes of fundamental importance when $Tr[G^2]$ is very large (e.g. for estimation protocols that are based on a multi-round procedures, where $\gamma \gg 1$) since as shown in (II.14) the first order correction in $\delta\lambda$ would be amplified by a factor γ^3. Furthermore, if the initial state is even slightly impure ($z = 1-\epsilon$, $\epsilon \ll 1$) the choice $\hat{b} = \hat{\alpha}$ becomes the only for which the bound can be fully attained. The above reasoning can be summarized as follows. On one hand the condition $\hat{b} = \hat{\alpha}$ guarantees that $F(\mathcal{B}_{\hat{b}}, |\psi_{\delta\lambda}\rangle\langle\psi_{\delta\lambda}|)$ has a maximum equal to the QFI i.e., it guarantees the highest sensitivity in the variation of λ. On the other hand, the condition $\hat{b} = \hat{\alpha}$ allows to have the lowest sensitivity with respect to small variations of the measurement angles $\delta\theta, \delta\phi$ and the purity $z \lesssim 1$.

III. SLD AND COHERENCE FOR N-DIMENSIONAL STATES

In the following we give the demonstration of result 3.1 in Proposition 3. We will use the following notations: given $|\alpha_{\pm,k}\rangle$, the eigenstates of L_0, we define the probabilities

$$p^{\lambda}_{\pm,k} = \langle\alpha_{\pm,k}|\rho_\lambda|\alpha_{\pm,k}\rangle, \quad p^{\lambda}_{\pm} = \sum_k p^{\lambda}_{\pm,k}, \quad p^{\lambda}_{k} = \sum_{i=\pm} p^{\lambda}_{i,k} \tag{III.1}$$

Under the following hypotheses:

- N is even;
- the initial diagonal state $\rho_0 = \sum_n p_n |n\rangle\langle n|$ is full rank
- $\langle n|G|m\rangle \in \mathbb{R}\ \forall n, m$ i.e., G has purely real matrix elements when expressed in the eigenbasis of ρ_0
- $L_{\lambda=0}$ is full rank.

Under the above hypotheses, it holds that:

1. the eigenvalues of L_0 are opposite in pairs,

$$L_0|\alpha_{\pm,k}\rangle = \pm\alpha_k|\alpha_{\pm,k}\rangle \tag{III.2}$$

and the Quantum Fisher Information reads

$$QFI \ = \ 2\sum_{k=1}^{N/2} (\alpha_{+,k})^2\, p^0_{+,k} \tag{III.3}$$

2. The coherence function of the eigenbasis $\mathcal{B}_{\pm,k} = \{|\alpha_{\pm,k}\rangle\}_{k=1}^{N/2}$ with respect to the state ρ_λ reads

$$Coh_{\mathcal{B}_{\pm,k}}(\rho_\lambda) = -V(\rho_\lambda) - 2\sum_k p^{\lambda}_{+,k}\log_2 p^{\lambda}_{+,k} \tag{III.4}$$

The Quantum Fisher Information is attained in correspondence of a critical point of $Coh_{\mathcal{B}_{\pm,k}}(\rho_\lambda)$ and

$$-\left[\partial^2 Coh_{\mathcal{B}_{\pm,k}}(\rho_\lambda)\right]_{\lambda=0} \ = \ QFI + f(\rho_{\lambda=0})$$

with

$$f(\rho_{\lambda=0}) \ = \ \sum_k \left(\partial^2_\lambda p^{\lambda}_k\right)_{\lambda=0} \log p^0_k$$

***Proof*.** We start by analyzing the eigendecomposition of L_0, the symmetric logarithmic derivative in $\lambda = 0$, and proving 1). L_0 is in the form[5]

$$\langle n|L_0|m\rangle = 2i\langle n|G|m\rangle(p_m - p_n)/(p_m + p_n)$$

Proof. When G is real in the eigenbasis of ρ_0, $\langle n|G|m\rangle \in \mathbb{R}$, then L_0 is purely imaginary ($L_0 = -L_0^*$), and thus it can be written as $L_0 = i\tilde{L}_0$, with $\tilde{L}_0$ real ($\langle n|\tilde{L}_0|m\rangle \in \mathbb{R}$) and antisymmetric ($\langle n|\tilde{L}_0|m\rangle = -\langle m|\tilde{L}_0|n\rangle$). Therefore, there exists a real orthogonal matrix O, $\langle n|O|m\rangle \in \mathbb{R}$ implementing a change of basis $|l_n\rangle = O|n\rangle$ such that $O^T\tilde{L}_0 O$ is in a standard form (see e.g. [76]), i.e., it is block diagonal and composed by $N/2$ blocks of dimension 2×2 of the form

$$\begin{pmatrix} 0 & -\alpha_k \\ \alpha_k & 0 \end{pmatrix} = -\alpha_k|l_{2k-1}\rangle\langle l_{2k}| + \alpha_k|l_{2k-1}\rangle\langle l_k|$$

Correspondingly, the SLD is in a block diagonal form $L_0 = \oplus_k \left(\alpha_k\sigma_y^k\right)$ in the basis of the $|l_n\rangle$. Each block can be diagonalized by the same kind of unitary transformation

$$\left(\mathbb{I} + i\sigma_x^k\right)/\sqrt{2} = \left[|l_{2k-1}\rangle\langle l_{2k-1}| + |l_{2k}\rangle\langle l_{2k}| + i\left(|l_{2k-1}\rangle\langle l_{2k}| + |l_{2k-1}\rangle\langle l_{2k}|\right)\right]/\sqrt{2}$$

i.e., the whole matrix can be diagonalized by means of the block-diagonal unitary operator $U = \frac{1}{\sqrt{2}}\oplus_k\left(\mathbb{I} + i\sigma_x^k\right)$. The eigenvectors of L_0 can be expressed as

$$|\alpha_{\pm,k}\rangle = (|l_{2k-1}\rangle \pm i|l_{2k}\rangle)/\sqrt{2} \tag{III.5}$$

with $k = 1, .., N/2$ and we obtain the result in Eq. (III.2),

$$L_0|\alpha_{\pm,k}\rangle = \pm\alpha_k|\alpha_{\pm,k}\rangle$$

The eigenvalues of L_0 are opposite in pairs, $\alpha_{\pm,k} = \pm\alpha_k,\ \alpha_k > 0$. We now show that

$$p^0_{+,k} = p^0_{-,k} \tag{III.6}$$

The reality of G implies that $[\rho_0, G]^T = -[\rho_0, G]$, thus the commutator is itself anti-symmetric. Since the change of basis O is real and it preserves the anti-symmetry of $[\rho_0, G]$, the diagonal elements of $[\rho_0, G]$ in the $\{|l_n\rangle\}$ basis are zero:

$$\langle l_{2k-1}|\,[\rho_0, G]\,|l_{2k-1}\rangle = 0$$

Therefore, taking into account that $[\rho_0, G] = -i(L_0\rho + \rho L_0)$, we also have

$$\langle l_{2k-1}|(L_0\rho + \rho L_0)|l_{2k-1}\rangle = 2Re\left\{\langle l_{2k-1}|L_0\rho_0|l_{2k-1}\rangle\right\} = 0$$

If we now express $|l_{2k-1}\rangle$ in terms of the respective $|\alpha_{\pm,k}\rangle$ we have that

$$2Re\left\{\langle l_{2k-1}|L_0\rho_0|l_{2k-1}\rangle\right\} \;=\; \alpha_+^k p^0_{+,k} - \alpha_+^k p^0_{-,k} = 0$$

since the "cross term"

$$\alpha_{+,k}\langle\alpha_{+,k}|\rho_0|\alpha_{-,k}\rangle/2 - \alpha_{+,k}\langle\alpha_{-,k}|\rho_0|\alpha_{+,k}\rangle/2 = i\alpha_{+,k}Im\left\{\langle\alpha_{+,k}|\rho_0|\alpha_{-,k}\rangle\right\}$$

is purely imaginary. Thus, we finally obtain the result in Eq. (III.6)

$$p^0_{+,k} = p^0_{-,k}$$

From this result one can easily derive some relations for the marginal probabilities $p_\pm = \sum_k p_{\pm,k}$ and $p_k = \sum_{i=\pm} p_{\pm,i}$. Since $\sum_{k,i=\pm} p^0_{i,k} = 2\sum_{k,i=\pm} p^0_{i,k} = 2p^0_+ = 1$, we get

$$p^0_+ = p^0_- = 1/2 \tag{III.7}$$

Moreover,

$$p_k = \sum_{i=\pm} p^0_{i,k} = 2p^0_{+,k} \tag{III.8}$$

From Eqs. (III.7) and (III.8) one also obtains that the probability distribution is factorized in $\lambda = 0$,

$$p^0_{+,k} = p^0_+ p^0_k \tag{III.9}$$

We are now ready to derive Eq. (III.3). Given $\Pi_{\pm,k} = |\alpha_{\pm,k}\rangle\langle\alpha_{\pm,k}|$, the derivatives of the $p^\lambda_{\pm,k}$ are

$$\begin{aligned} \left(\partial_\lambda p^\lambda_{\pm,k}\right)_{\lambda=0} &= Tr\left\{\Pi_{\pm,k}\left(\partial_\lambda \rho^\lambda\right)_{\lambda=0}\right\} = iTr\left\{\Pi_{\pm,k}\left[\rho_0, G\right]\right\} = \\ = Re\left\{Tr[\rho_0 \Pi_{\pm,k} L_0]\right\} &= Re\left\{\pm\alpha^+_k Tr[\rho_0 \Pi_{\pm,k}]\right\} = \pm\alpha_{+,k} p^0_{\pm,k} \end{aligned}$$

so that

$$\left(\partial_\lambda p^\lambda_{\pm,k}\right)_{\lambda=0} = \pm\alpha_{+,k} p^0_{\pm,k} \tag{III.10}$$

and the QFI reads

$$\begin{aligned} QFI &= \sum_{i=\pm,k=1}^{N/2} \left(\partial_\lambda p^\lambda_{\pm,k}\right)^2_{\lambda=0} / p^0_{i,k} = \\ &= 2\sum_{k=1}^{N/2} \left(\partial_\lambda p^\lambda_{+,k}\right)^2_{\lambda=0} / p^0_{+,k} = \\ &= 2\sum_{k=1}^{N/2} \left(\alpha_{+,k}\right)^2 p^0_{+,k} \end{aligned}$$

where between the first and the second line we have used Eq. (III.8) and from the second to the third line Eq. (III.10).

We now prove 2). The coherence function $Coh_{\mathcal{B}_{\pm,k}}(\rho_\lambda)$ reads by definition

$$Coh_{\mathcal{B}_{\pm,k}}(\rho_\lambda) = -V(\rho_\lambda) - \sum_{k,i=\pm} p^\lambda_{i,k} \log_2 p^\lambda_{i,k}$$

By considering Eqs (III.6) and (III.10), one obtains

$$\begin{aligned} \left(\partial_\lambda Coh_{\mathcal{B}_{\pm,k}}(\rho_\lambda)\right)_{\lambda=0} &= -\sum_{\pm,k} \left(\partial_\lambda p^\lambda_{\pm,k}\right)_{\lambda=0} \log\left(p^0_{\pm,k}\right) \\ &= -\sum_k \alpha_{+,k} p^0_{+,k} \left(\log\left(p^0_{+,k}\right) - \log\left(p^0_{-,k}\right)\right) = 0 \end{aligned}$$

i.e., the coherence function for the basis $\mathcal{B}_{\pm,k}$ has a critical point in $\lambda = 0$.

$$\begin{aligned} \left[\partial^2_\lambda Coh_{\mathcal{B}_{\pm,k}}(\rho_\lambda)\right]_{\lambda=0} &= -\sum_{i=\pm,,k} \frac{\left(\partial_\lambda p^\lambda_{i,k}\right)^2_{\lambda=0}}{p^0_{i,k}} - \sum_{i=\pm,k} \left(\partial^2_\lambda p^\lambda_{i,k}\right)_{\lambda=0} \log\left(p^0_{i,k}\right) \\ &= -QFI - \sum_{,k} \left(\left(\partial^2_\lambda p^\lambda_{+,k}\right)_{\lambda=0} + \left(\partial^2_\lambda p^\lambda_{-,k}\right)_{\lambda=0}\right) \log\left(p^0_{+,k}\right) \\ &= -QFI - \sum_k \left(\partial^2_\lambda p^\lambda_k\right)_{\lambda=0} \log p^0_{+,k} \\ &= -QFI - \sum_k \left(\partial^2_\lambda p^\lambda_k\right)_{\lambda=0} \log p^0_k \end{aligned}$$

where: in the second last line we have used $p^0_{+,k} = p^0_{-,k}$; in the third line $\left(\partial^2_\lambda p^\lambda_{+,k}\right)_{\lambda=0} + \left(\partial^2_\lambda p^\lambda_{-,k}\right)_{\lambda=0} = \left(\partial^2_\lambda \left(p^\lambda_{+,k} + p^\lambda_{-,k}\right)\right) = \left(\partial^2_\lambda p^\lambda_k\right)_{\lambda=0}$; in the last line we have used $p^0_{+,k} = p^0_+ p^0_{,k} = p^0_k/2$ and $\sum_k \left(\partial^2_\lambda p^\lambda_k\right)_{\lambda=0} \log 2 = 0$. □

IV. SLD-INDUCED TPS FOR N-DIMENSIONAL STATES

When is even N, G has real matrix elements in the eigenbasis of ρ_0 and L_0 is full rank the eigenstates of the SLD provide a natural way to introduce a proper tensor product structure TPS^R that allows to relate the QFI to a specific kind of classical correlations.

We first show how the result 3.2) in Proposition 3 can be derived.

Given two subalgebras $\mathcal{A}_A, \mathcal{A}_B \subset End(\mathcal{H})$ they induce a tensor product structure [53] if the following conditions are satisfied: i) independence, $[\mathcal{A}_A, \mathcal{A}_B] = 0$ ii) completeness, $\mathcal{A}_A \vee \mathcal{A}_B = End(\mathcal{H})$. In our case, given the $N/2$ pairs of eigenstates of the SLD

$$|\alpha_{\pm,k}\rangle = \left(|l_{2k-1}\rangle \pm i|l_{2k}\rangle\right)/\sqrt{2},\;\; k = 1, 2, .., N/2$$

one can identify a new TPS that will split the Hilbert in a "qubit" and an $N/2$-dimensional space, $\mathcal{H}_N \sim \mathcal{H}_2 \tilde{\otimes} \mathcal{H}_{N/2}$ and will allow writing $|\alpha_{\pm,k}\rangle = |\pm\rangle \tilde{\otimes} |k\rangle$. The subalgebras of Hermitian operators $\mathcal{A}_2, \mathcal{A}_{N/2}$ acting locally on $\mathcal{H}_2$ and $\mathcal{H}_{N/2}$ are identified as follows. We choose $\mathcal{A}_2 = span\,\{\sigma_0, \sigma_x, \sigma_y, \sigma_z\} \cong u(2)$ where

$$\begin{aligned}
\sigma_x &\equiv \sum_{k=1}^{N/2} \left(|l_{2k-1}\rangle\langle l_{2k}| + |l_{2k}\rangle\langle l_{2k-1}|\right) = \sum_{k=1}^{N/2} \left(|\alpha_{+,k}\rangle\langle\alpha_{-,k}| + |\alpha_{-,k}\rangle\langle\alpha_{+,k}|\right) \\
\sigma_y &\equiv -i\sum_{k=1}^{N/2} \left(|l_{2k-1}\rangle\langle l_{2k}| - |l_{2k}\rangle\langle l_{2k-1}|\right) = \sum_{k=1}^{N/2} \left(|\alpha_{+,k}\rangle\langle\alpha_{+,k}| - |\alpha_{-,k}\rangle\langle\alpha_{-,k}|\right) \\
\sigma_z &\equiv \sum_{k=1}^{N/2} \left(|l_{2k-1}\rangle\langle l_{2k-1}| - |l_{2k}\rangle\langle l_{2k}|\right) = i\sum_{k=1}^{N/2} \left(|\alpha_{+,k}\rangle\langle\alpha_{-,k}| - |\alpha_{-,k}\rangle\langle\alpha_{+,k}|\right) \\
\sigma_0 &\equiv \sum_{k=1}^{N/2} \left(|\alpha_{+,k}\rangle\langle\alpha_{+,k}| + |\alpha_{-,k}\rangle\langle\alpha_{-,k}|\right) = \sum_{k=1}^{N/2} \left(|\alpha_{+,k}\rangle\langle\alpha_{+,k}| + |\alpha_{-,k}\rangle\langle\alpha_{-,k}|\right)
\end{aligned} \tag{IV.1}$$

The other subalgebra $\mathcal{A}_{N/2} \cong u(N/2)$ can be constructed in an analogous way by starting from the following general definition of the operators that form a basis of $u(N/2)$

$$\mathcal{A}_{N/2} \equiv span\,\{|k\rangle\langle h| + |h\rangle\langle k|, \quad -i|k\rangle\langle h| + i|h\rangle\langle k|, \quad |h\rangle\langle h|,\; h \neq k = 1, \ldots, N/2\}$$

where, in order to adapt the result to our specific case one has to use

$$|k\rangle\langle h| \equiv |l_{2k-1}\rangle\langle l_{2h-1}| + |l_{2k}\rangle\langle l_{2h}| = |\alpha_{+,k}\rangle\langle\alpha_{+,h}| + |\alpha_{-,k}\rangle\langle\alpha_{-,h}|, \qquad k \neq h \in 1, 2, .., N/2$$

$$|h\rangle\langle h| \equiv |l_{2h-1}\rangle\langle l_{2h-1}| + |l_{2h}\rangle\langle l_{2h}| = |\alpha_{+,h}\rangle\langle\alpha_{+,h}| + |\alpha_{-,h}\rangle\langle\alpha_{-,h}| \qquad h \in 1, 2, .., N/2 \tag{IV.2}$$

One has that $\left[\mathcal{A}_2, \mathcal{A}_{N/2}\right] = 0$, $\mathcal{A}_1 \vee \mathcal{A}_2 = u(N)$ and therefore these subalgebras identify a well-defined TPS $\mathcal{H}_N \sim \mathcal{H}_2 \tilde{\otimes} \mathcal{H}_{N/2}$, correspondingly the SLD eigenvectors can be written as

$$|\alpha_{\pm,k}\rangle = |\pm\rangle \tilde{\otimes} |k\rangle \tag{IV.3}$$

In the new TPS, we can write the operators in (IV.1) as

$$\sigma_x = S_x \tilde{\otimes} \mathbb{I}_{N/2} \qquad \sigma_y = S_y \tilde{\otimes} \mathbb{I}_{N/2} \qquad \sigma_z = S_z \tilde{\otimes} \mathbb{I}_{N/2} \tag{IV.4}$$

$$\sigma_0 = \mathbb{I}_2 \tilde{\otimes} \mathbb{I}_{N/2} \tag{IV.5}$$

where S_x, S_y, S_z are Pauli operators acting on the single qubit factor $\mathcal{H}_2$. The operators in (IV.2) can be written as

$$|k\rangle\langle h| \to \mathbb{I}_2 \tilde{\otimes} |k\rangle\langle h|, \qquad |h\rangle\langle h| \to \mathbb{I}_2 \tilde{\otimes} |h\rangle\langle h| \tag{IV.6}$$

and they for a basis for the Hermitian operators acting on $\mathcal{H}_{N/2}$. For all $O_2 \in \mathcal{A}_2, O_{N/2} \in \mathcal{A}_{N/2}$ the composition of the operators in $\mathcal{H}_N$ is given by $O_2 O_{N/2}$ that now can be written as $O_2 O_{N/2} \simeq O_2 \tilde{\otimes} O_{N/2}$; onto the basis states one has $O_2 O_{N/2} |\alpha_{\pm,k}\rangle = O_2 |\pm\rangle \otimes O_{N/2} |k\rangle$.

Before passing to the rest of the proof, we notice that even if the full controllability of the single $End(\mathcal{H}_2), End(\mathcal{H}_{N/2})$ is not practically at hand, in order to carry over the estimation process one needs only to be able to implement the measurement process identified by $\mathcal{B}_{\pm,k}$, which amounts to experimentally observing the probabilities $p_{\pm,k} = Tr\left[|\alpha_{\pm,k}\rangle\langle\alpha_{\pm,k}|\rho\right] = Tr\left[\Pi_{\pm} \otimes \Pi_k \rho\right]$ i.e., the joint probabilities of an experiment carried over onto the entire $\mathcal{H}_N$. And thus the probabilities pertaining to the local observables $\left(\Pi_{\pm} \otimes \mathbb{I}_{N/2}, \mathbb{I}_2 \otimes \Pi_k\right)$ i.e., the marginals $p_{\pm}, p_k$, can be easily derived.

The SLD in the TPS^R can be written in terms of the new product basis $\mathcal{B}_{\alpha=\pm,\mathbf{k}} = \left\{|\pm\rangle\tilde{\otimes}|k\rangle\right\}$ as

$$L_0 = \sum_{k=1,..,N/2} \alpha_{+,k}\left(\Pi_+\tilde{\otimes}\Pi_k - \Pi_-\tilde{\otimes}\Pi_k\right)$$

where

$$\begin{aligned} \Pi_+\tilde{\otimes}\mathbb{I}_{N/2} &\equiv \sum_{k=1}^{N/2} |\alpha_{+,k}\rangle\langle\alpha_{+,k}| = && \text{(IV.7)} \\ &= \left[\mathbb{I}_N + \sum_{k=1}^{N/2}\left(|\alpha_{+,k}\rangle\langle\alpha_{+,k}| - |\alpha_{-,k}\rangle\langle\alpha_{-,k}|\right)\right]/2 && \text{(IV.8)} \\ &= \frac{(\mathbb{I}_2 + S_y)}{2}\tilde{\otimes}\mathbb{I}_{N/2} && \text{(IV.9)} \end{aligned}$$

where in the second line we have used $\sum_{k=1}^{N/2}|\alpha_{+,k}\rangle\langle\alpha_{+,k}| = \mathbb{I}_N - \sum_{k=1}^{N/2}|\alpha_{-,k}\rangle\langle\alpha_{-,k}|$. Analogously

$$\Pi_-\tilde{\otimes}\mathbb{I}_{N/2} = \frac{(\mathbb{I}_2 - S_y)}{2}\tilde{\otimes}\mathbb{I}_{N/2}.$$

On the other hand

$$\mathbb{I}_2 \otimes \Pi_k \equiv \sum_{i=\pm}|\alpha_{i,k}\rangle\langle\alpha_{i,k}| = \mathbb{I}_2\tilde{\otimes}|k\rangle\langle k| \tag{IV.10}$$

Given the previous definitions, the probabilities defined in the previous section read

$$\begin{aligned} p^{\lambda}_{\pm,k} &= \langle\alpha_{\pm,k}|\rho_\lambda|\alpha_{\pm,k}\rangle = Tr[\Pi_\pm \otimes \Pi_k\rho_\lambda] \\ p^{\lambda}_{\pm} &= \sum_{k=1}^{N/2}\langle\alpha_{\pm,k}|\rho_\lambda|\alpha_{\pm,k}\rangle = Tr[\Pi_\pm \otimes \mathbb{I}_{N/2}\rho_\lambda] \\ p^{\lambda}_{k} &= \sum_{i=\pm}\langle\alpha_{i,k}|\rho_\lambda|\alpha_{i,k}\rangle = Tr[\mathbb{I}_2 \otimes \Pi_k\rho_\lambda] \end{aligned}$$

and they correspond to an experiment with joint $\left(\Pi_\pm\tilde{\otimes}\Pi_k\right)$ vs local $\left(\Pi_\pm\tilde{\otimes}\mathbb{I}_{N/2}\right), \left(\mathbb{I}_2\tilde{\otimes}\Pi_k\right)$ measurements onto ρ_λ. In general, the set of probabilities $p^{\lambda}_{\pm,k}, p^{\lambda}_{\pm}, p^{\lambda}_{,k}$ are *those generated by the measurement of any observable O commuting with* L_0 onto ρ_λ. And the correlations relative to those observables can be expressed by the mutual information.

We are now ready to derive result 4.3 in Proposition 4. Given the definition of mutual information

$$\mathcal{M}^{\lambda}_{L_0} \equiv \mathcal{H}(p^{\lambda}_{\pm}) + \mathcal{H}(p^{\lambda}_{k}) - \mathcal{H}(p^{\lambda}_{\pm,k}) \tag{IV.11}$$

one has from Eq. (III.9) that $p^0_{\pm,k} = p^0_\pm p^0_k$ and therefore

$$\mathcal{M}^{0}_{L_0} = \mathcal{H}(p^{0}_{\pm}) + \mathcal{H}(p^{0}_{k}) - \mathcal{H}(p^{0}_{\pm,k}) = 0$$

i.e., the observables that commute with L_0 are uncorrelated. In terms of the probabilities the coherence function can be written as

$$Coh_{\mathcal{B}_{\pm,k}}(\rho_\lambda) = -\mathcal{V}(\rho_\lambda) + \mathcal{H}(p^{\lambda}_{\pm,k}) \tag{IV.12}$$

or alternatively

$$Coh_{\mathcal{B}_{\pm,k}}(\rho_\lambda) = -\mathcal{V}(\rho_\lambda) + \mathcal{H}(p^{\lambda}_{\pm}) + \mathcal{H}(p^{\lambda}_{k}) - \mathcal{M}^{\lambda}_{L_0}. \tag{IV.13}$$

If now one computes the $\left[\partial^2_\lambda Coh_{\mathcal{B}_{\pm,k}}(\rho_\lambda)\right]_{\lambda=0}$, from (IV.12) one has that

$$\left[\partial^2 Coh_{\mathcal{B}_{\pm,k}}(\rho_\lambda)\right]_{\lambda=0} \;=\; -\sum_k \left(\partial^2 p_k^\lambda\right)_{\lambda=0} \log p_k^0 - QFI$$

while from (IV.13)

$$\begin{aligned}\left[\partial^2 Coh_{\mathcal{B}_{\pm,k}}(\rho_\lambda)\right]_{\lambda=0} \;&=\; -\sum_i \frac{\left(\partial p_i^\lambda\right)^2_{\lambda=0}}{p_i^0} - \sum_k \left(\partial^2 p_k^\lambda\right)_{\lambda=0} \log p_k^0 + \\ &- \;\left(\partial^2 \mathcal{M}^\lambda\right)_{\lambda=0}\end{aligned}$$

The results can be obtained using the above found relations (III.6), (III.9), and (III.10), for the probabilities and their derivatives in $\lambda = 0$. Equating the previous two expression for the second order derivative of the coherence one obtains

$$\begin{aligned} QFI \;&=\; FI_2 + \left(\partial^2_\lambda \mathcal{M}^\lambda_{L_0}\right)_{\lambda=0} \\ &=\; \sum_i \frac{\left(\partial_\lambda p_i^\lambda\right)^2_{\lambda=0}}{p_i^0} + \left(\partial^2_\lambda \mathcal{M}^\lambda_{L_0}\right)_{\lambda=0}\end{aligned}$$

i.e., the result (25) in the main text. According to the latter, the QFI is composed by two contributions. The first term is the single qubit Fisher Information FI_2 that one would obtain by measuring $\Pi_\pm$ onto the single qubit reduced density matrix $\xi_\lambda = Tr_{N/2}\left[\rho_\lambda\right]$. The second term is given by the second order variation of the mutual information $\mathcal{M}^\lambda_{L_0}$ between the relevant observables O that commute with the SLD L_0. Since we now that $\mathcal{M}^{\lambda=0}_{L_0} = 0$, the point $\lambda = 0$ is a minimum for $\mathcal{M}^\lambda_{L_0}$ and therefore $\left(\partial^2_\lambda \mathcal{M}^\lambda_{L_0}\right)_{\lambda=0} > 0$.

V. N-DIMENSIONAL MIXED STATES MAXIMAL QFI

Suppose $\rho_0 = \sum_n p_n |n\rangle\langle n|$ is diagonal and the p_n are in *decreasing order*. The QFI for general N-dimensional mixed states reads

$$QFI = 2\sum_{i\neq j} \frac{(p_i - p_j)^2}{(p_i + p_j)} |G_{ij}|^2 = 4\sum_{i<j} \frac{(p_i - p_j)^2}{(p_i + p_j)} |G_{ij}|^2$$

Proposition 5. *The problem of optimizing the QFI over all G such that $Tr[G^2] = \sum_{ij} |G_{ij}|^2 \leq 2\gamma^2$ has the following solution:*

$$\max_{Tr[G^2]\leq g^2} QFI = 4\gamma^2 \frac{(p_1 - p_N)^2}{(p_1 + p_N)}$$

where the optimal G has $|G_{1N}| = |G_{N1}| = \gamma$ and all the remaining $|G|_{ij} = 0$ (including G_{ii}).

Proof. Since the QFI only depends on the off-diagonal terms of G when represented in the ρ_0 eigenbasis, the optimization can be done by considering operators G such that, in the same basis, $G_{ii} = 0,\ i = 1, ..N$. Then the optimization problem can be written as follows:

$$\max \sum_{k=1}^M a_k x_k \quad \text{over} \sum_{k=1}^M x_k \leq \gamma^2, x_k > 0$$

where $M = N(N-1)/2$; $\{a_k\} = \left\{\frac{(p_i - p_j)^2}{(p_i + p_j)}, i < j\right\}$ for $1 \leq k \leq M$; and $\{x_k\} = \{|G_{ij}|^2, i < j\}$ for $1 \leq k \leq M$. This is a simple *linear program*[77]. The optimal solution is found on a vertex of the feasible region defined by $\sum_{k=1}^M x_k \leq \gamma^2, x_k > 0$. The vertices are the M points $v_1 = \{x_1 = \gamma^2, 0, \ldots, 0\}, \ldots, v_M = \{0, \ldots, 0, x_M = \gamma^2\}$. The maximum is the found at v_ℓ where $a_\ell = \max a_k$ and it is unique if $\max a_k$ is unique. We then have

$$\max_{Tr[G^2]\leq g^2/2} QFI = 4\gamma^2 \max_{ij} \left(\frac{(p_i - p_j)^2}{(p_i + p_j)}\right)$$

It can be easily seen that $\max_{ij}\left(\frac{(p_i-p_j)^2}{(p_i+p_j)}\right) = \frac{(p_1-p_N)^2}{(p_1+p_N)}$. Indeed for each pair i,j with one has $\frac{(p_i-p_j)^2}{(p_i+p_j)} = p_i\frac{(1-x)^2}{(1+x)}$ where $x = p_j/p_i$ and we assume (without restriction of generality) that $p_i > p_j$. Now, $\frac{(1-x)^2}{(1+x)}$ is a monotonically decreasing function of x, so it attains it maximum for the minimum x, given by p_N/p_1. Then, since $p_i \le p_1$, we have $\frac{(p_i-p_j)^2}{(p_i+p_j)} \le \frac{(p_1-p_N)^2}{(p_1+p_N)}$. □ □

Let us now assume that the dimension N is even. The optimal G is $G_{1N} = \gamma$, which corresponds to $G = \gamma\sigma_x$ in the $|1\rangle, |N\rangle$ subspace. We have

$$L_0 = i\gamma\frac{(p_1 - p_N)}{(p_1 + p_N)}(|1\rangle\langle N| - |N\rangle\langle 1|)$$

The eigenvalues of L_0 are

$$\alpha_{\pm,1} = \pm\gamma\frac{(p_1 - p_N)}{(p_1 + p_N)}, \alpha_{\pm,k} = 0 \ \forall k = 2, \dots, N/2$$

As for the optimal measurement basis $\mathcal{B}_{\boldsymbol{\alpha}} = \{|\alpha_{\pm,1}\rangle\} \bigcup \{|\alpha_{\pm,k}\rangle\}_{k=2}^{N/2}$ one has that

$$|\alpha_{\pm,1}\rangle = \frac{1}{\sqrt{2}}(|1\rangle \mp i|N\rangle)$$

while since the kernel of L_0 has dimension $N-2$, one has a lot of freedom in the choice of remaining part of the basis $\{|\alpha_{\pm,k}\rangle\}_{k=2}^{N/2}$. Whatever the choice of $\mathcal{B}_{\boldsymbol{\alpha}}$ one has

$$p^{\lambda}_{\pm,1} = \frac{p_1 + p_N}{2} \pm \frac{p_1 - p_N}{2}\sin(2\lambda\gamma)$$

and since $p^{\lambda}_{\pm,k}$ are independent of λ for any $k \ge 2$, it follows that

$$[\partial_\lambda Coh_{\mathcal{B}_{\boldsymbol{\alpha}}}(\rho_\lambda)]_{\lambda=0} = \sum_{i=\pm}\left(\partial_\lambda p^{\lambda}_{i,1}\right)_{\lambda=0}\log p^0_{i,1} = 0$$

i.e., the coherence has a critical point in $\lambda = 0$. Moreover, since $\left(\partial^2_\lambda p^{\lambda}_{\pm,k}\right)_{\lambda=0} = 0 \ \forall k$, we get

$$-\left[\partial^2_\lambda Coh_{\mathcal{B}_{\boldsymbol{\alpha}}}(\rho_\lambda)\right]_{\lambda=0} = QFI = 4g^{Bures}_\lambda \tag{V.1}$$

Therefore the QFI is identically equal to the second order variation of $Coh_{\mathcal{B}_{\boldsymbol{\alpha}}}(\rho_\lambda)$. As for the decomposition of QFI (25) will vary depending on the choice of the kernel's basis, since the value of $p^0_\pm = \sum_k p^0_{\pm,k}$depends on the actual choice and

$$FI_2 = \left(\partial_\lambda p^{\lambda}_{+,1}\right)^2_{\lambda=0}\left(\frac{1}{p^0_+\left(1-p^0_+\right)}\right)$$

$$\left(\partial^2_\lambda \mathcal{M}^{\lambda}_{L_0}\right)_{\lambda=0} = QFI - FI_2 = \left(\partial_\lambda p^{\lambda}_{+,1}\right)^2_{\lambda=0}\left(\frac{2}{p^0_{+,1}} - \frac{1}{p^0_+\left(1-p^0_+\right)}\right)$$

Since $p^0_+ \ge p^0_{+,1}$, and $p^0_+\left(1-p^0_+\right) \le 1/4$ one has

$$4\left(\partial_\lambda p^{\lambda}_{+,1}\right)^2_{\lambda=0} \le FI_2 \le \left(\partial_\lambda p^{\lambda}_{+,1}\right)^2_{\lambda=0}\left(\frac{1}{p^0_{+,1}\left(1-p^0_{+,1}\right)}\right) = QFI$$

$$\left(\partial_\lambda p^{\lambda}_{+,1}\right)^2_{\lambda=0}\left(\frac{2}{p^0_{+,1}} - 4\right) \ge \left(\partial^2_\lambda \mathcal{M}^{\lambda}_{L_0}\right)_{\lambda=0} \ge 0$$

For all bases $\mathcal{B}_{\boldsymbol{\alpha}}$ one always has $\left(\partial^2_\lambda \mathcal{M}^{\lambda}_{L_0}\right)_{\lambda=0} > 0$. In particular, if the choice is such that $p^0_\pm = 1/2$ one has that the single qubit contribution FI_2 is minimal while $\left(\partial^2_\lambda \mathcal{M}^{\lambda}_{L_0}\right)_{\lambda=0}$ is maximal. For example one can choose for the kernel of L_0 the basis

$$|\alpha_{\pm,k}\rangle \;=\; \frac{1}{\sqrt{2}}(|2k-2\rangle \pm |2k-1\rangle)\; k=2,\ldots,N/2$$

where $|2k-2\rangle, |2k-1\rangle$ $k=2,\ldots,N/2$ are eigenstates of ρ_0. Accordingly one can define the TPS^R (IV.7) $\mathcal{H} \sim \mathcal{H}_2\tilde{\otimes}\mathcal{H}_{N/2}$. With respect with such representation the state reads

$$\rho = \sum_k p_k(\mathbb{I}_2 + h_k S_z)\tilde{\otimes}|k\rangle\langle k|$$

with $h_k = \frac{p_{2k-2}-p_{2k-1}}{p_{2k-2}+p_{2k-1}}, k>1$ and $h_1 = \frac{p_1-p_N}{p_1+p_N}$. On the other hand $G = S_x\tilde{\otimes}|1\rangle\langle 1|$ i.e., G acts as a conditional rotation on the single qubit. The probabilities for measurement in the defined $|\alpha_{\pm,k}\rangle$ basis are

$$p^\lambda_{\pm,1} = \frac{p_1+p_N}{2} \pm \frac{p_1-p_N}{2}\sin 2\lambda\gamma, \qquad p^\lambda_{\pm,k} = \frac{p_{2k-2}+p_{2k-1}}{2}, \quad k=2,\ldots,N/2$$

Thus, one obtains

$$p^\lambda_\pm = \frac{1}{2} \pm \frac{p_1-p_N}{2}\sin 2\gamma\lambda, \qquad p^\lambda_{k=1} = p_1+p_N, \qquad p^\lambda_{k>1} = p_{2k-2}+p_{2k-1}$$

from which we have $\partial_\lambda p^\lambda_\pm = \pm 2\gamma \frac{p_1-p_N}{2}\cos\lambda\gamma$ and finally

$$FI_2 = \sum_{i=\pm} \frac{\left(\partial_\lambda p^\lambda_i\right)^2_{\lambda=0}}{p^0_i} = 4\gamma^2(p_1-p_N)^2 = (p_1+p_N)QFI$$

$$\left(\partial^2_\lambda \mathcal{M}^\lambda_{L_0}\right)_{\lambda=0} = 4\gamma^2(p_1-p_N)^2\left(\frac{1-p_1-p_N}{p_1+p_N}\right) = QFI(1-p_1-p_N)$$

i.e., the result reported in the main text. The (maximal) value $\left(\partial^2_\lambda \mathcal{M}^\lambda_{L_0}\right)_{\lambda=0}$ vanishes in the limit of $p_1 \to 1$ and $p_n \to 0,\ \forall n>1$ i.e., in the limiting case of a pure state.

Class of separable states

Consider the (separable but generally discordant) states

$$\rho_0 = \sum_{k=1}^N p_k \tau_k \otimes |k\rangle\langle k| \tag{V.2}$$

where $\tau_k = (\mathbb{I} + \vec{n}_k\cdot\vec{\sigma})/2$ are pure states in the xy plane, $\vec{n}_k = (\cos\delta_k, \sin\delta_k, 0)$, and $G = \sigma_z \otimes \mathbb{I}_{N/2}$. The SLD reads $L_0 = \oplus_k L_k$ with $L_k = 2\hat{\alpha}_k\cdot\vec{\sigma}$ and $\hat{\alpha}_k = \hat{n}_k \times \hat{z}$. The eigenvectors of L_0 are

$$|\alpha_{\pm,k}\rangle = |\pm\hat{\alpha}_k\rangle \otimes |k\rangle$$

where $|\pm\hat{\alpha}_k\rangle$ are the states corresponding to the Bloch vectors $\pm\hat{\alpha}_k\cdot\vec{\sigma}$. The TPS^R construction, which allows writing $|\pm\hat{\alpha}_k\rangle \otimes |k\rangle = |\pm\rangle\tilde{\otimes}|k\rangle$ is nontrivial. However, as for applying Proposition 4, one needs only to compute the joint marginal and probabilities for an experiment in the SLD eigenbasis. We find

$$\begin{aligned} p^\lambda_{\pm,k} &= \frac{1}{2}(1\pm\sin 2\lambda)p_k \\ p^\lambda_\pm &= \frac{1}{2}(1\pm\sin 2\lambda) \end{aligned}$$

from which we obtain $p^\lambda_{\pm,k} = p^\lambda_\pm p^0_k$ such that $\mathcal{M}^\lambda_{L_0} = 0$ for all λ and

$$QFI \;=\; \sum_{i=\pm,k}\left(\partial_\lambda p^\lambda_{i,k}\right)^2_{\lambda=0}/p^0_{i,k} = \sum_k p_k \sum_{i=\pm}\left(\partial_\lambda p^\lambda_i\right)^2_{\lambda=0}/p^0_i = \sum_k p_k QFI_k = \sum_{i=\pm}\left(\partial_\lambda p^\lambda_i\right)^2_{\lambda=0}/p^0_i = 4$$

The overall estimation precision is a weighted (in terms of the p_k) sum of single qubit estimation precisions $QFI_k = \sum_{i=\pm} \left(\partial_\lambda p_i^\lambda\right)^2_{\lambda=0} / p_i^0,\ \forall k$. What matters is the variation of the coherence

$$-\left[\partial^2_\lambda Coh_{|\alpha_{\pm,k}\rangle}(\tau_k)\right]_{\lambda=0} = QFI_k$$

enacted by G for each single qubit state τ_k. The above analysis holds for any generic state of the class (V.2); therefore it holds for any dimension N, for whatever probability distribution $\{p_k\}$ and thus for whatever value of the discord between the subsystem $\mathcal{H}_2$ and $\mathcal{H}_{N/2}$.

If now for example one of the $\vec{n}_h = (\cos\delta_h, 0, \sin\delta_h)$ does not lie in the xy plane, the state has still discord, and the above analysis still holds except that now for the specific h the $QFI_h = 4\sin^2\delta_h < 4$ and thus the overall QFI decreases with respect to the previous case.
As specific illustrative example of the above reasoning we choose the following two-qubit states:

$$\rho_1 = \left(|0_x\rangle\langle 0_x| \otimes |0_x\rangle\langle 0_x| + |1_x\rangle\langle 1_x| \otimes |1_x\rangle\langle 1_x|\right)/2$$

$$\rho_2 = \left(|0_x\rangle\langle 0_x| \otimes |0_x\rangle\langle 0_x| + |1_z\rangle\langle 1_z| \otimes |1_x\rangle\langle 1_x|\right)/2$$

where $|0_{x,z}\rangle$, $|1_{x,z}\rangle$ are an eigenstates of $\sigma_{x,z}$. While ρ_1 has discord zero, ρ_2 has discord different from zero. As generator of the phase shift we choose $G = \sigma_z \otimes \mathbb{I}_2$. The estimation is a single qubit one and the overall QFI is equal to 4 for ρ_1, while it is equal to 2 for ρ_2. Notice that presence of discord is not detrimental per se; it is detrimental for the estimation procedure because, such kind of quantum correlations are due to the presence of $|1\rangle_{zz}\langle 1|$ in ρ_2 which however does not contribute to the estimation process.

GHZ state

The definition of the TPS^R has been explicitly given in the main text. The eigenvectors of L_0 are

$$|\pm\rangle\tilde{\otimes}|k\rangle \ = \ \left(|GHZ_k^+\rangle \pm i|GHZ_k^-\rangle\right)/\sqrt{2}$$

We first write the operator $G = \sum_h \sigma_z^h$ in TPS^R. Each σ_z^h acts on M-qubits states o*f the computational basis* $\{|k\rangle_M = |k_M, .., k_1\rangle\}$ as:

$$\sigma_z^h|k\rangle_M = (-1)^{k_h}|k\rangle_M$$

where k_h is the h-th digit of the binary representation of k. One has

$$\sigma_z^h|\pm\rangle\tilde{\otimes}|k\rangle \ = \ (-1)^{k_h}(\pm i)|\mp\rangle$$

and therefore σ_z^h it can be represented within the k-th sector as $(-1)^{k_h} S_x \otimes \Pi_k$ and on the overall state space as

$$\sigma_z^h = S_x\tilde{\otimes}\sum_k (-1)^{k_h}\Pi_k \tag{V.3}$$

Consequently the whole Hamiltonian acts as

$$\sum_h \sigma_z^h = S_x\tilde{\otimes}\sum_k\left[\left(\sum_h (-1)^{k_h}\right)\Pi_k\right] \tag{V.4}$$

where $\sum_h (-1)^{k_h} = M - 2|k|$ is the difference between the number of zeros $M - |k|$ and the number of ones $|k|$ present in the M digits binary representation of k. Therefore over the whole state

$$G = \sum_h \sigma_z^h \ = \ S_x\tilde{\otimes}\sum_k (M - 2|k|)\Pi_k \tag{V.5}$$

The action of $U_\lambda = \exp - i\lambda G$ onto the initial state $\rho_0 = \sum_k p_k|GHZ_k^+\rangle\langle GHZ_k^+| = |0\rangle_{zz}\langle 0|\tilde{\otimes}\sum_k p_k\Pi_k$ gives

$$\rho_\lambda \ = \ \sum_k p_k\tau_k^\lambda\tilde{\otimes}\Pi_k.$$

with $\tau_k^\lambda = e^{-i\lambda(M-2|k|)S_x}|0\rangle_{zz}\langle 0|e^{i\lambda(M-2|k|)S_x}$. In each sector k the state τ_k^λ is pure and its Bloch vector is given by $(0, \sin(2\lambda(M-2|k|)), \cos(2\lambda(M-2|k|)))$. Therefore the measurement onto the eigenstates of $S_y \otimes \Pi_k$ in each sector k

$$p_{\pm,k}^\lambda = \frac{1}{2}\left[1 \pm \sin 2\lambda(M-2|k|)\right] p_k$$

such that $p_{\pm,k}^0 = p_k/2$ and since

$$p_{\pm,}^\lambda = \frac{1}{2}\sum_k \left[1 \pm \sin 2\lambda(M-2|k|)\right] p_k$$

one has $p_\pm^0 = 1/2$. Furthermore

$$\left(\partial_\lambda p_{\pm,k}^\lambda\right)_{\lambda=0} = \pm(M-2|k|)p_k.$$

and QFI therefore is given by

$$QFI = \sum_k (M-2|k|)^2 p_k$$

Furthermore from

$$\left(\partial_\lambda p_{\pm,}^\lambda\right)_{\lambda=0} = \pm\sum_k (M-2|k|)p_k.$$

one gets

$$FI_2 = 4\left(\sum_k (M-2|k|)p_k\right)^2.$$

VI. QFI AND COHERENCE FOR THE GHZ STATE UNDER NOISE

Noise map and its action on the GHZ state.

The solution of the master equation (45) was given in Ref.[66] and we report it here for the sake of completeness. The single-qubit map $\Lambda_{\gamma,\omega}$ can be written in Kraus form as $\Lambda_{\gamma,\omega}(\rho) = \sum_{i,j=\{0,x,y,z\}} S_{ij}\sigma_i\rho\sigma_j$ with $S_{00} = a+b$, $S_{xx} = d+f$, $S_{yy} = d-f$, $S_{zz} = a-b$, $S_{0z} = S_{z0}^* = ic$ with

$$\begin{aligned}
a &= e^{-\gamma/2t}\cosh\gamma t \\
b &= e^{-\gamma/2t}\cos(\zeta_{\omega,\gamma}t) \\
c &= 2\omega/\zeta_{\omega,\gamma}e^{-\gamma/2t}\sin(\zeta_{\omega,\gamma}t) \\
d &= e^{-\gamma/2t}\sinh\gamma t \\
f &= \gamma/\zeta_{\omega,\gamma}e^{-\gamma/2t}\sin(\zeta_{\omega,\gamma}t)
\end{aligned}$$

with $\zeta_{\omega,\gamma} = \sqrt{4\omega^2-\gamma^2}$.

As shown in Ref.[66], acting on each qubit of the GHZ state $\rho_0 = |GHZ_0^+\rangle\langle GHZ_0^+|$ with

$$|GHZ_0^\pm\rangle = \left(|00\ldots0\rangle \pm |11\ldots1\rangle\right)/\sqrt{2}$$

the map yields a state $\rho_{\omega,\gamma}(t)$ that is block-diagonal with 2-dimensional blocks. Indeed, the only nonzero off-diagonal elements are

$${}_M\langle k|\rho_{\omega,\gamma}(t)|\bar{k}\rangle_M = \left({}_M\langle\bar{k}|\rho_{\omega,\gamma}(t)|k\rangle_M\right)^*$$

where $|k\rangle_M \equiv |k_M,..,k_1\rangle$, $|\bar{k}\rangle_M \equiv |\bar{k}_M,..,\bar{k}_1\rangle$, $k=0,..,2^{M-1}-1$ is the computational basis of the global Hilbert space. One has

$$ {}_M\langle k|\rho_{\omega,\gamma}(t)|\bar{k}\rangle_M = \frac{1}{2}\left[f^{|k|}(b-ic)^{M-|k|} + f^{M-|k|}(b+ic)^{|k|}\right] $$

where $|k|$ is the number of ones in the string $k_1 \dots k_M$, while the diagonal elements are

$$ {}_M\langle k|\rho_{\omega,\gamma}(t)|k\rangle_M = \frac{1}{2}\left[d^{|k|}a^{M-|k|} + a^{M-|k|}d^{|k|}\right] ={}_M \langle\bar{k}|\rho_{\omega,\gamma}(t)|\bar{k}\rangle_M $$

As a result, the state can be written as

$$ \rho_{\omega,\gamma}(t) = \sum_k r_k \left(|k\rangle_{MM}\langle k| + |\bar{k}\rangle_{MM}\langle\bar{k}|\right) + \left(s_k|k\rangle_{MM}\langle\bar{k}| + h.c.\right) \tag{VI.1} $$

with $r_k ={}_M \langle k|\rho_{\omega,\gamma}(t)|k\rangle_M$ and $s_k ={}_M \langle k|\rho_{\omega,\gamma}(t)|\bar{k}\rangle_M$.

TPS^R notation

In the following we explicitly develop the calculations that allow to write Eqs. (50), (55), (56). We first start by writing the state $\rho_{\omega,\gamma}(t)$ in the TPS^R corresponding to the noiseless case, see (31). The basis states are

$$ \left(|GHZ_k^+\rangle \pm i|GHZ_k^-\rangle\right)/\sqrt{2} \;=\; \left((1\pm i)|k\rangle_M + (1\mp i)|\bar{k}\rangle_M\right)/2 = \tag{VI.2} $$

$$ = \; |\pm\rangle\tilde{\otimes}|k\rangle \tag{VI.3} $$

where now $\mathcal{H}_{2^{M-1}} = span\{|k\rangle\}$. The initial state of the evolution is $|GHZ_0^+\rangle = \frac{(|+\rangle+|-\rangle)}{\sqrt{2}}|0\rangle$ while the state (VI.1) can be written as

$$ \rho_{\omega,\gamma}(t) = \sum_k p_k(t)\tau_k(\omega,t)\tilde{\otimes}|k\rangle\langle k| $$

with

$$ p_k(t)\tau_k(\omega,t) = \begin{pmatrix} r_k + Re(s_k) & -i\; Im(s_k) \\ i\; Im(s_k) & r_k - Re(s_k) \end{pmatrix} $$

Parallel noise

We now exploit the description of $\rho(\omega,t)$ and G (V.5)in TPS^R in order to write the coherent part of the evolution (46) as

$$ -i\frac{\omega}{2}\left[\sum_h \sigma_z^h, \rho(\omega,t)\right] \;= $$

$$ -i\frac{\omega}{2}\sum_k \left[S_x, \tau_k(\omega,t)\right]\tilde{\otimes}\left(N-2|k|\right)\Pi_k. \tag{VI.4} $$

where $\tilde{\tau}_k(\omega,t)$ is the un-normalized single qubit state pertaining to the sector k, each of which enjoys a coherent dynamics described by

$$ -i\frac{\omega}{2}\left(N-2|k|\right)\left[S_x, \tau_k(\omega,t)\right]. \tag{VI.5} $$

We now focus on the decoherent part of the master equation (47) for the case of parallel noise i.e., $\alpha_z = 1, \alpha_x = \alpha_y = 0$ and $\mathcal{L}(\rho) = -\frac{\gamma}{2}\sum_h\left[\rho - \sum_h \sigma_z^h\rho\sigma_z^h\right]$. Given the representation of σ_z^h operators (V.3), one finds that

$$ \sum_h \sigma_z^h\rho\sigma_z^h \;=\; \sum_k S_x\left[p_k(t)\tau_k(\omega,t)\right]S_x\tilde{\otimes}\sum_h (-1)^{2k^h}\,\Pi_{k_M} = $$

$$ = \; M\sum_k S_x\left[p_k(t)\tau_k(\omega,t)\right]S_x\tilde{\otimes}\Pi_k \tag{VI.6} $$

since $\sum_h (-1)^{2k^h} = M$ for all k's. Therefore, in the parallel noise case the decoherent part does not couple the various sectors k. This together with the fact that the initial state is $\frac{(|+\rangle+|-\rangle)}{\sqrt{2}}|0\rangle$ shows that the noisy evolution takes place in the $k=0$ sector only. By using (VI.4) and (VI.6) the master equation reduces to the single differential equation for the single qubit state τ_0 reported in the main text i.e.,

$$\begin{aligned}\partial_t \tau_0 &= -\frac{iM\omega}{2}[S_x, \tau_0] + \\ &- \frac{M\gamma}{2}[\tau_0 - S_x \tau_0 S_x]\end{aligned}$$

Transverse noise.

In order to describe the representation of the master equation in the case of transverse noise in the above introduced TPS, we first give the representation of σ_x^h. For $h < M$, the latter acts onto the computational basis states as:

$$\sigma_x^h |k\rangle_M = |k_M, .., k_{h+1}, \bar{k}_h, k_{h-1}, ..k_1\rangle \equiv |k'(h)\rangle_M$$

with $k'(h) \in [0, \dots, 2^{M-1}-1]$ and analogously

$$\sigma_x^h |\bar{k}\rangle_M = |\bar{k}_M, .., \bar{k}_{h+1}, k_h, \bar{k}_{h-1}, ..\bar{k}_1\rangle \equiv |\overline{k'(h)}\rangle_M$$

where k_h is the h-th digit of the binary representation of k and $\bar{k}_h$ its negated value. We have $k'(h) = k + (-1)^{k_h} 2^{h-1}$ and the number of ones in the binary representation of $k'(h)$ is given by $|k'(h)| = |k| + (-1)^{k_h} = |k| \pm 1$. Therefore σ_x^h, $h < M$ has the effect of a permutation of the k sectors.

$$\sigma_x^h |\pm\rangle \tilde{\otimes} |k\rangle = |\pm\rangle \tilde{\otimes} |k(h)\rangle$$

For $h = M$, one gets

$$\sigma_x^M |k\rangle_M = |\bar{k}_M, k_{M-1}.., k_1\rangle \equiv |\bar{k}'(M)\rangle_M$$

$$\sigma_x^M |\bar{k}\rangle_M = |k_M, \bar{k}_{M-1}, ..\bar{k}_1\rangle \equiv |k'(M)\rangle_M$$

such that

$$\sigma_x^M |\pm\rangle \tilde{\otimes} |k\rangle = |\mp\rangle \tilde{\otimes} |k'(M)\rangle$$

Here, $k'(M) = 2^{M-1} - k - 1$ and the number of ones in the binary representation of $k'(M)$ is given by $|k'(M)| = M - 1 - |k|$. Each k sector is coupled, by means of σ_x^h's, to the sectors $k'(h)$. The representation of σ_x^h in the TPS^R is for $h < M$

$$\sigma_x^h = \mathbb{I}_2 \tilde{\otimes} O_h \qquad \text{(VI.7)}$$

where the traceless unitary operator $O_h = (\sum_{k=0} |k'(h)\rangle\langle k| + |k\rangle\langle k'(h)|) = O_h^\dagger$ enacts a permutation on the basis states $|k\rangle$, while

$$\sigma_x^M = S_z \tilde{\otimes} O_M \qquad \text{(VI.8)}$$

with $O_M = \sum_{k=0} |k'(M)\rangle\langle k| + |k\rangle\langle k'(M)|$.

Let us check how decoherence works for a single ρ_k. As we have already seen the coherent part of the evolution can be written as

$$\begin{aligned}Tr\left[\mathbb{I}_2 \tilde{\otimes} \Pi_k \left(\frac{-i\omega}{2}[H, \rho(\omega, t)]\right)\right] &= \\ = \frac{-i\omega(M - 2|k|)}{2}[S_x, \tilde{\tau}_k].\end{aligned}$$

If one instead takes the whole trace $Tr_{\mathcal{H}_{2M-1}}$, one has to sum up the last relation for all k obtaining

$$\frac{-i\omega}{2}\sum_{k}\left(M-2|k|\right)\left[S_{x},\tilde{\tau}_{k}\right] \tag{VI.9}$$

As for the decoherent part we have to first write the term $\sigma_x^h\rho\sigma_x^h$. One has that

$$\begin{aligned} O_h\Pi_k O_h^{\dagger} &= \Pi_{k'(h)} \\ O_h\Pi_{k'(h)} O_h^{\dagger} &= \Pi_k \end{aligned}$$

and therefore for each $h<M$

$$\begin{aligned} \tilde{\tau}_k(\omega,t)\tilde{\otimes}\Pi_k &\rightarrow \tilde{\tau}_k(\omega,t)\tilde{\otimes}\Pi_{k'(h)} \\ \tilde{\tau}_{k'(h)}(\omega,t)\tilde{\otimes}\Pi_{k'(h)} &\rightarrow \tilde{\tau}_{k'(h)}(\omega,t)\tilde{\otimes}\Pi_k \end{aligned}$$

Therefore, one gets

$$\sigma_x^h\rho\sigma_x^h = \sum_k \tilde{\tau}_k(\omega,t)\tilde{\otimes}\Pi_{k'(h)} = \sum_k \tilde{\tau}_{k'(h)}(\omega,t)\tilde{\otimes}\Pi_k$$

The effect is therefore to reshuffle the original state upon exchanging $\tilde{\tau}_k(\omega,t)\leftrightarrow\tilde{\tau}_{k'(h)}(\omega,t)$. For $h=M$ one has

$$\begin{aligned} \sigma_x^M\rho\sigma_x^M &= \sum_k S_z\tilde{\tau}_{k'(M)}S_z\tilde{\otimes}\Pi_k \\ \sigma_x^M\rho\sigma_x^M &= \sum_k S_z\tilde{\tau}_k S_z\tilde{\otimes}\Pi_{k'(M)} \end{aligned}$$

The effect is thus to reshuffle the original state upon exchanging $\tilde{\tau}_k(\omega,t)\leftrightarrow\tilde{\tau}_{k'(M)}(\omega,t)$ and apply a S_z rotation. Taking everything into account, the differential equation for a single $\tilde{\tau}_k(\omega,t)$ can be written as

$$\partial_t\tilde{\tau}_k(\omega,t) = -\frac{i\omega\left(M-2|k|\right)}{2}\left[S_x,\tilde{\tau}_k\right]-\frac{\gamma}{2}\left(M\tilde{\tau}_k-\left[S_z\tilde{\tau}_{k'(M)}S_z+\sum_{h=1}^{M-1}\tilde{\tau}_{k'(h)}\right]\right)$$

To obtain the evolution of the reduced state $\xi\left(t,\omega\right)$, one should take the trace over $\mathcal{H}_{2^{M-1}}$ which just corresponds to summing over k. Since $\sum_k\tilde{\tau}_k(\omega,t)=\xi\left(t,\omega\right)$ but also $\sum_k\tilde{\tau}_{k'(h)}=\xi\left(t,\omega\right)$, and $\sum_{k'}\left(\sum_{h=1}^{M-1}\tilde{\tau}_{k'(h)}\right)=(M-1)\,\xi\left(t,\omega\right)$ we finally get

$$\partial_t\xi\left(t,\omega\right) = \frac{-i\omega}{2}\left[S_x,\sum_k\left(M-2|k|\right)\tilde{\tau}_k\right]+\frac{\gamma}{2}\left(\xi\left(t,\omega\right)-S_z\xi\left(t,\omega\right)S_z\right)$$

VII. COHERENCE AND QPTS

We consider a family of states $|0_\lambda\rangle$ that are the ground states of the generic Hamiltonian $H_\lambda=H_0+\lambda V$ labeled by a continuous parameter λ. In the same notation of Methods xx, we can write, to first order in $\delta\lambda$, $|0^{\lambda+\delta\lambda}\rangle=|0^\lambda\rangle+\delta\lambda|v\rangle$ where $|v\rangle$ is the first order correction one can obtain with standard perturbative analysis [78]:

$$|v\rangle=|v^\perp\rangle=\sum_{n\neq0}\frac{\langle0^\lambda|V|n^\lambda\rangle}{(E_n^\lambda-E_0^\lambda)}|n^\lambda\rangle$$

with $|n^\lambda\rangle, E_n^\lambda$ eigenvectors and eigenvalues of H_λ. It holds $\langle v|v\rangle=\sum_{n\neq0}\frac{\left|\langle0^\lambda|V|n^\lambda\rangle\right|^2}{\left(E_n^\lambda-E_0^\lambda\right)^2}$ and we define $|\hat{v}\rangle=|v\rangle/\sqrt{\langle v|v\rangle}$; by construction $\langle0^\lambda|\hat{v}\rangle=0$. As eigenbasis of the SLD we can choose $\mathcal{B}_{\boldsymbol{\alpha}}=\left\{|\alpha_\pm\rangle=\frac{1}{\sqrt{2}}(|0^\lambda\rangle\pm|v\rangle)\right\}\bigcup\{|2\rangle,\dots|N\rangle\}$

with the only requirement that $\langle \alpha_\pm | n \rangle = 0 \ \forall n \geq 2$. By again using the same notations of Methods xx we obtain the measurement probabilities

$$p_\pm^{\lambda+\delta\lambda} = \left| \left\langle 0^{\lambda+\delta\lambda} | \alpha_\pm \right\rangle \right|^2 = \frac{1}{2}\left(1 \pm 2|v|\delta\lambda\right) + \mathcal{O}(\delta\lambda^2)$$

and

$$p_n^\lambda = \left| \left\langle 0^{\lambda+\delta\lambda} | n \right\rangle \right|^2 = \mathcal{O}(\delta\lambda^3) \ \forall n \geq 2$$

.Consequently we obtain the desired result

$$\begin{aligned} QFI &= -\left(\partial^2_{\delta\lambda} Coh_{\mathcal{B}_\alpha}\right)_{\delta\lambda=0} \\ &= \sum_{i=\pm} \frac{\left(\partial_{\delta\lambda} p_i^{\delta\lambda}\right)_{\delta\lambda=0}}{p_i^0} = 4|v|^2 = \\ &= 4 \sum_{n\neq 0} \frac{|\langle 0^\lambda | V | n^\lambda \rangle|^2}{(E_n^\lambda - E_0^\lambda)^2} = 4 g_\lambda^{FS}. \end{aligned}$$

Notice that, although the choice of $\{|2\rangle, \ldots |N\rangle\}$ is not unique the result holds for any of the possible choices as long as $\langle \alpha_\pm | n \rangle = 0 \ \forall n \geq 2$. The scaling properties when $\lambda \to \lambda_c$ follow from the those of g_λ^{FS}.