Role of Initial Entanglement and Non-Gaussianity in the Decoherence of Photon-Number Entangled States Evolving in a Noisy Channel

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We address the degradation of continuous variable (CV) entanglement in a noisy channel focusing on the set of photon-number entangled states. We exploit several separability criteria and compare the resulting separation times with the value of non-Gaussianity at any time, thus showing that in the low-temperature regime: (i) non-Gaussianity is a bound for the relative entropy of entanglement and (ii) Simon's criterion provides a reliable estimate of the separation time also for non-Gaussian states. We provide several evidences supporting the conjecture that Gaussian entanglement is the most robust against noise, i.e., it survives longer than a non-Gaussian one, and that this may be a general feature for CV systems in Markovian channels.

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Continuous variable (CV) quantum information has been developed with Gaussian states and operations [1–3]. However, in the recent years also the non-Gaussian sector of the Hilbert space has been taken into consideration. This interest is due to the potential role of non-Gaussianity in enhancing long-distance quantum communication based on entanglement distillation [4,5] and swapping, quantum memories [6], cloning [7] and teleportation [8]. In turn, it has become of interest to analyze non-Gaussian states in realistic conditions [9], where decoherence due to dissipation and thermal noise unavoidably leads to degradation of entanglement. Our work is indeed motivated by the following general question: in case of transmission through a noisy environment is there any advantage in using non-Gaussian states? Do they lose entanglement in a longer time? We provide evidence for the answer to be negative, thus supporting the conjecture that Gaussian entanglement is extremal in terms of robustness against decoherence due to noise and dissipation.

In order to address the above questions, in this Letter we consider a broad and meaningful class of CV bipartite states endowed with perfect correlations in the number of photons: photon-number entangled states (PNES). The latter have Schmidt decomposition in the Fock basis, i.e.,

$$|\psi\rangle = \sum_{n=0}^{\infty} \psi_n |n\rangle|n\rangle \tag{1}$$

with real coefficients $\psi_n \in \mathbb{R}$, $\psi_n > 0$, $\sum_{n=0}^{\infty} \psi_n^2 = 1$. The advantages of considering these states are twofold. They are sufficiently simple for analytical study, and at the same time meaningful since several experimental realizations have been reported [10] and quantum communication schemes involving PNES have been proposed [11]. Furthermore, the set of PNES contains mostly non-Gaussian states but includes (as a subclass) two-mode squeezed vacua, i.e., the basic Gaussian resource for CV quantum information, thus allowing for a direct comparison between Gaussian and non-Gaussian states. Finally, PNES are good candidates for long-distance quantum communication, because they have been already proved robust against some kind of noise, e.g., phase diffusion [12]. We consider several special subclasses of PNES with specific parametric dependence, as well as randomly generated [13] (truncated) PNES, in order to draw some general conclusions about the typical behavior of entanglement dynamics. In particular, we focus on random PNES with decreasing profile (i.e., $\psi_n > \psi_{n+1}$) and on the following parametric subclasses (we omit normalization): (i) the two-mode squeezed vacua or twin-beam states (TWB) $\psi_n \propto x^n 0 \leq x < 1$ which are the sole Gaussian states within the PNES class and represent the preferred (Gaussian) resources in protocols involving CV entanglement; (ii) the photon subtracted (PSSV) [14] $\psi_n \propto (n+1)x^{n+1}$ and the photon-added two-mode squeezed vacua (PASV) [15] $\psi_n \propto n x^{n-1}$, which are obtained from the TWB by the experimentally feasible operations of photon subtraction $\varrho \rightarrow a_1 a_2 \varrho a_1^\dagger a_2^\dagger$ and addition $\varrho \rightarrow a_1^\dagger a_2^\dagger \varrho a_1 a_2$ respectively [16]; (iii) the pair-coherent or two-mode coherently correlated states (TMC) [17] with Poissonian profile $\psi_n \propto \frac{\lambda^n}{n!}$, $\lambda \in \mathbb{R}$. The mean energy of PNES is $\langle\psi|a_1^\dagger a_1 + a_2^\dagger a_2|\psi\rangle = 2N$, where $N = \sum_{n=0}^{\infty} |\psi_n|^2 n$, whereas correlations between the modes can be quantified by $C = \mathrm{Re}\sum_{n=0}^{\infty} \psi_n^* \psi_{n+1}(n+1)$ and entanglement is given by the Von Neumann entropy of the partial traces $\epsilon_0 = -\sum_n \psi_n^2 \log \psi_n^2$. In turn, the covariance matrix (CM) of a PNES equals that of a symmetric Gaussian state in standard form, with diagonal elements equal to $N + \frac{1}{2}$ and off-diagonal blocks given by $\boldsymbol{C} = \mathrm{diag}(C, -C)$.

The propagation in noisy channels can be modeled as the interaction of the two modes with two independent thermal

baths of oscillators. The resulting dynamics is a Gaussian channel, governed by the two–mode Master equation (ME) $\dot{\varrho} = \sum_{j=1,2} \frac{\Gamma}{2} N_j L[a_j^\dagger]\varrho + \frac{\Gamma}{2}(N_j + 1)L[a_j]\varrho$ describing losses and thermal hopping in presence of (local) non–classical fluctuations of the environment. Dot stands for time-derivative and the Lindblad superoperator is defined by $L[O]\varrho \equiv 2O\varrho O^\dagger - O^\dagger O\varrho - \varrho O^\dagger O$. Γ is a loss coefficient and N_j are the mean photon numbers in the stationary state, which is a thermal state. We consider baths at equal temperature $N_1 = N_2 = N_T$. The above ME admits the operator solution [18]: $\varrho(t) = \Lambda_t \varrho(0) = \mathrm{Tr}_{34}[U_t(\varrho(0) \otimes \nu_{34})U_t]$, where Λ_t denotes the evolution map corresponding to the noisy channel; 3,4 are two additional fictitious modes in a thermal state $\nu_{34} = \nu_3 \otimes \nu_4$; $U_t = U_{13}(\zeta_t) \otimes U_{24}(\zeta_t)$ and $U_{ij}(\zeta_t) = \exp(\zeta_t a_i^\dagger a_j - \zeta_t^* a_j^\dagger a_i)$ is the two-mode mixing operator, with $\zeta_t = \arctan(e^{\Gamma t} - 1)^{1/2}$. Using this solution, the evolved density matrix ϱ_t can be computed numerically from the initial state ϱ_0 upon truncating the Hilbert space dimension. In our study we consider states with total energy $0 \le 2N \le 10$ and dimension $D = 20$. In this range of energies, and for all subclasses of states, the adopted truncation results in a negligible error. We emphasize that the map Λ_t, being the product of two local maps, can only disrupt quantum correlations: for any $N_T \neq 0$ we have a complete loss of entanglement within a finite, state dependent, time $t_S = t_S(\varrho)$ which we refer to as the *separation time*.

In order to estimate t_S for non-Gaussian states subjected to the action of Λ_t we make use of several entanglement criteria and this also enables a comparison of their performances in detecting entanglement. As it is well known, in the CV case a necessary-and-sufficient separability criterion exists only for Gaussian states [19]: Simon's criterion (SI) for separability is equivalent to the positivity of the partial transpose density matrix and says that a Gaussian state is separable iff $\tilde{d}_- < 1/2$, where $\tilde{d}_-$ is the least symplectic eigenvalue of the CM of the partial-transposed state. When dealing with CV non-Gaussian states, Simon's criterion (which is equivalent to the separability of a Gaussian state having the same CM as the given state) is only sufficient for entanglement. This actually holds for any available criterion: if the state is entangled, a given test may or not detect its entanglement; in turn, if no test detects entanglement, we can not conclude separability of the state. The Simon separation time t_{SI} can be computed analytically. At the level of CM, the map Λ_t induces the evolution $\sigma_t = \sigma_0 e^{-\Gamma t} + \sigma_\infty(1 - e^{-\Gamma t})$, where $\sigma_\infty = \mathrm{diag}(N_T + 1/2, \ldots, N_T + 1/2)$ is the asymptotic thermal state CM. The CM of the partial-transposed state is given by $\Lambda\sigma_t\Lambda$, where $\Lambda = \mathrm{diag}(1, 1, 1, -1)$ [19], and we have $\tilde{d}_- = (N_T + 1/2)(1 - e^{-\Gamma t}) + (N + 1/2)e^{-\Gamma t} - |C|e^{-\Gamma t}$. Therefore, for $N_T = 0$ PNES are entangled at any time, whereas for $N_T \neq 0$ we have a lower bound to separability $t_{\mathrm{SI}} = \frac{1}{\Gamma}\log(1 + \frac{|C|-N}{N_T})$.

Besides Simon's criterion, we will make use of three different criteria which provide independent separability conditions. The first is the extension of SI given by Shchukin and Vogel [20] (SH) based on the evaluation of a series of $M \times M$ matrices whose entries are moments up to a given order: non-positive definiteness of any finite submatrix is a sufficient condition for entanglement. By considering the minor defined by the first and second-order moments only ($M = 5$) we obtain a condition which is equivalent to SI. If we consider larger minors, moments of higher order are involved and we get a stronger condition. Here we consider moments up to order 8. The second criterion has been introduced by Sperling and Vogel (SP) [21] and it is based on linear entanglement witnesses. A state ϱ is entangled if $\langle\phi|\varrho|\phi\rangle > \max_n\{|m_n|^2\}$, where $|\phi\rangle$ is a pure entangled state with Schmidt coefficients $\{m_n\}$. We test this condition by using 10^4 randomly generated witnesses of the form $|\phi\rangle = \sum_{n=1}^{D} \phi_n|n\rangle|n\rangle$ with $D = 20$, i.e., the witnesses are themselves truncated random PNES. This form is chosen since the bath does not create quantum correlations but only destroys those originally present. Finally, the realignment criterion [22] (RE) is based on positivity of a linear contraction map: a state ϱ is entangled if $\|\tilde{\varrho}\| > 1$ where $\|A\|$ denotes the trace norm of operator A and $\langle m|\langle\mu|\tilde{\varrho}|n\rangle|\nu\rangle = \langle m|\langle n|\varrho|\nu\rangle|\mu\rangle$. Using these criteria, we obtain lower bounds on separation times. Indeed, for any given criterion K and state ϱ, let us denote by $t_K(\varrho)$ the maximum time for which K proves that ϱ is entangled: clearly $t_K(\varrho)$ is a lower bound for t_S. Considering the best bound we have $t_S(\varrho) \ge t_M = \max_K t_K(\varrho)$.

The propagation in noisy channels, besides entanglement, also destroys the non-Gaussian character of the initial state, which unavoidably evolves towards the asymptotic, Gaussian thermal state. We shall take into account both processes (separation and Gaussification) in parallel and explore the relations between them. The non-Gaussian character of a state ϱ is measured by $\delta(\varrho) = S(\tau) - S(\varrho)$ i.e, the relative entropy between ϱ and the reference Gaussian state τ having its same covariance matrix [23]. In order to explore the effect of noise in a wide range of conditions and initial states we consider TWB, PSSV, PASV, TMC, and random PNES of different energies and compute the evolved density matrix for $0 \le t \le 15$ in units of inverse loss $1/\Gamma$. At any time t, entanglement is tested with all the above mentioned criteria and the value of the non-Gaussianity δ is computed. From these data we evaluate t_K, i.e., lower bounds to separation times according to different criteria, and Gaussification times t_G, i.e., times for which non-Gaussianity δ falls below a fixed Gaussification threshold δ_G (we consider different thresholds $\delta_G = 0.1, 0.01, 0.001$). The procedure is then repeated for different values of the temperature T corresponding to N_T in the range $[10^{-5}, 10^{-1}]$.

We start describing the results of our analysis by focusing on TMC. In Fig. 1 we report t_K for TMC and different criteria as a function of N for the lowest (highest) tempera-

ture considered $N_T = 10^{-5}$ (10^{-1}). It turns out that at any temperature SI, SH, and RE criteria yield similar curves whereas the SP criterion works only at low T. We point out two general features: (i) $\bar{t}_K$ is a decreasing function of T, i.e., entanglement is strongly corrupted as the temperature increases; (ii) both at high and low T, t_K rapidly increases to an asymptotic value $\bar{t}_K$ which is reached at $N \sim 1/2$ and then remains almost constant. In Fig. 1 we also show Gaussification times of TMC as a function of energy. We see that the behavior of non-Gaussianity is only weakly affected by the increase of T. Upon comparing separation and Gaussification times we notice that at low T states become nearly Gaussian well before they become separable: $t_G < t_K \leq t_S$. At high T, on the contrary, Gaussification times are greater than our bounds on separation times: $t_G > t_K$. The analysis of all other PNES subclasses (TWB, PASV, PSSV and random PNES) reveals the same qualitative behavior described for TMC for the evolution of both entanglement and non-Gaussianity (with the obvious exception of TWB, whose non-Gaussianity is always zero). In summary, we have numerically proved that for the whole class of states we have considered and at any temperature SI, SH, and RE criteria yield qualitatively the same results. In addition, Simon's criterion, which offers analytical advantages, is the optimal one. Furthermore, the separation time decreases with T, and the dependence on the energy can be appreciated only for small N, while they quickly reach their asymptotic values as N increases. As for the non-Gaussianity, we have $t_G < t_K \leq t_S$ at low T and $t_G > t_K$ at high T. This deserves further consideration. Indeed, the relation $t_G < t_K \leq t_S$ suggest that for low T the bounds provided by Simon's criterion properly estimate the actual PNES separation times, i.e., $t_{\mathrm{SI}} \simeq t_S$. This can be understood by first noticing that when $t > t_G$ the states are nearly Gaussian and therefore Simon's criterion is expected to be very reliable. Furthermore, at any $t \geq t_{\mathrm{SI}} > t_G$ the reference Gaussian state is obviously separable and thus the non-Gaussianity can be compared with a measure of entanglement: the relative entropy [24] $E(\varrho) = \min_{\sigma\in\Omega}[S(\varrho||\sigma)]$ that quantifies the distance between ϱ and the whole set of separable states Ω. When $t \geq t_{\mathrm{SI}}$ one has that

$$E(\varrho) \leq \delta(\varrho) \ll 1 \tag{2}$$

and this confirms that in this limit the states are very poorly entangled (if they are) and SI allows to reliably estimate t_S. The fact that the other criteria provide very close bounds on t_S strengthens our conclusion. At high T Gaussification times are greater than all t_K and we cannot draw the same conclusions. However, the agreement between different criteria is still an indication that t_K may represent a good estimate of t_S.

We now focus our attention on the dependence on t_{SI} on the initial non-Gaussianity δ_0. Indeed, we have $\delta_0 = 2f(d_-)$ where $f(x) = (x + 1/2)\log(x + 1/2) - (x - 1/2)\log(x - 1/2)$ monotonically increases with x [3] and $d_- = \sqrt{(N + 1/2)^2 - |C|^2}$ is the least symplectic eigenvalue of the covariance matrix. Upon defining $g = f^{-1}$, t_{SI} can be written as

$$t_{\mathrm{SI}} = \frac{1}{\Gamma}\log\left(1 + \frac{\sqrt{(N + 1/2)^2 - g^2(\delta_0/2)} - N}{N_T}\right), \tag{3}$$

which shows that t_{SI} is a decreasing function of δ_0 at any fixed N, and it is maximized by TWB for which $\delta_0 = 0$.

In the left panel of Fig. 2 the separation times and the initial non-Gaussianities of different PNES are plotted against N (for t_S we use $t_M = \max_K t_K(\varrho)$): at any fixed energy N, the states with higher δ_0 have shorter separation times. This result holds $\forall\, T$ and it is related with the fact that at any fixed N Gaussian states are maximally entangled [3]. Let us now discuss the relation between separation times, non-Gaussianity, and the initial entanglement of the states. As shown in the right panel of Fig. 2, where t_M and δ_0 are plotted as a function of the initial entanglement ϵ_0, the dependence is by no means universal. However, we notice that also at fixed ϵ_0 states with higher δ_0 show shorter t_S: this trend is not represented by an exact relation, but it represents a clear indication that non-Gaussianity speeds up the loss of entanglement, making Gaussian entanglement more robust than non-Gaussian one. Therefore the robustness of Gaussian entanglement may be conjectured to be a general feature of CV systems evolving in noisy Markovian channels. In fact, within the Markovian approximation, propagation in CV noisy channels corresponds to a ME in Linblad form, which induces a Gaussian map and enforces Gaussification of any initial state.

The results of our analysis, together with the above discussions, naturally lead us to formulate the following general conjecture: for any fixed value of the global energy of a PNES, and for any given noisy Markovian evolution with losses and thermal hopping, the Gaussian states are those that have maximal separation times. Besides, from

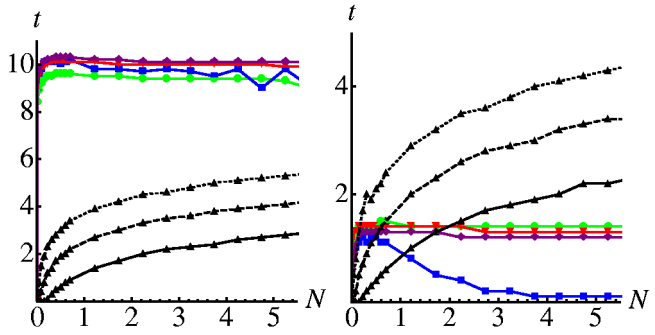

FIG. 1 (color online). Separation and Gaussification times for TMC as a function of the mean energy for low ($N_T = 10^{-5}$, left) and high ($N_T = 10^{-1}$, right) temperature. In both plots we report separation times according to different criteria: t_{RE} (green, circle), t_{SP} (blue, square), t_{SH} (red, triangle), t_{SI} (purple, rhombus), and Gaussification times for different thresholds: $\delta_G = 10^{-1}$ (solid black, triangle), $\delta_G = 10^{-2}$ (dashed black, triangle), $\delta_G = 10^{-3}$ (dotted black, triangle).

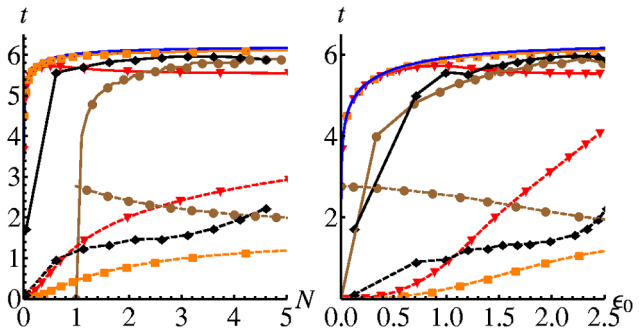

FIG. 2 (color online). Separation times t_S (straight lines) and initial non-Gaussianities δ_0 (dashed lines) as a function of initial energy N (left) and initial entanglement ϵ_0 (right) for different PNES classes: PASV (brown, circle), PSSV (orange, square), TMC (red, triangle), TWB (blue) and random (black, rhombus) states. In both cases the bath has fixed $N_T = 10^{-3}$.

Fig. 2 we also extract another relevant feature: in the high-energy limit there is an approximate universality in separation times, i.e., t_S are nearly constant and similar for all classes of states, including randomly generated states, independent of the non-Gaussianity: the effect of the departure from Gaussianity is very small.

Let us summarize the results of our analysis. We have considered a class of states (PNES) including Gaussian and non-Gaussian subclasses and exploited several entanglement criteria to estimate their separation times in a noisy channel. The analysis shows that no criterion is able to give better bounds than those provided by Simon's criterion. At low temperature, the estimate provided by Simon's criterion is very reliable since PNES Gaussify well before they lose entanglement, whereas at high temperature it represents a lower bound on separation time. At any fixed energy N, separation times decrease with the initial non-Gaussianity δ_0, both at high and at low temperature, whereas for any fixed initial entanglement ϵ_0 separation time is longer for states with lower δ_0, i.e., Gaussian entanglement is the most robust against noise. Finally, in the high-energy limit and independently of the temperature, the differences among separation times of different subclasses are small, non-Gaussian entanglement being nearly as robust as Gaussian one.

In conclusion, we have provided several evidences supporting the conjecture that at fixed energy Gaussian entanglement is the most robust against noise in a Markovian Gaussian channel. On the other hand, our analysis shows that robustness of non-Gaussian states is comparable with that of Gaussian states for sufficiently high energy of the states. This implies that in these regimes non-Gaussian resources can be exploited to improve quantum communication protocols approximately over the same distances.

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